THE BRAUER LOOP SCHEME AND ORBITAL VARIETIES

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ABSTRACT. A. Joseph invented multidegrees in [Jo84] to study **orbital varieties**, which are the components of an **orbital scheme**, itself constructed by intersecting a nilpotent orbit with a Borel subalgebra. Their multidegrees are known as **Joseph polynomials**, and these polynomials give a basis of a (Springer) representation of the Weyl group. In the case of the nilpotent orbit $\{M^2 = 0\}$, the orbital varieties can be indexed by noncrossing chord diagrams in the disc.

In this paper we study the *normal cone* to the orbital scheme inside this nilpotent orbit $\{M^2 = 0\}$. This gives a better-motivated construction of the *Brauer loop scheme* we introduced in [KZJ07], whose components are indexed by all chord diagrams (now possibly with crossings) in the disc.

The multidegrees of its components, the *Brauer loop varieties*, were shown to reproduce the ground state of the *Brauer loop model* in statistical mechanics [DFZJ06]. Here, we reformulate and slightly generalize these multidegrees in order to express them as solutions of the rational quantum Knizhnik–Zamolodchikov equation associated to the Brauer algebra. In particular, the vector of the multidegrees satisfies two sets of equations, corresponding to the e_i and f_i generators of the Brauer algebra. The proof of the analogous statement in [KZJ07] was slightly roundabout; we verified the f_i equation using the geometry of multidegrees, and used algebraic results of [DFZJ06] to show that it implied the e_i equation. We describe here the geometric meaning of both e_i and f_i equations in our slightly extended setting.

We also describe the corresponding actions at the level of orbital varieties: while only the e_i equations make sense directly on the Joseph polynomials, the f_i equations also appear if one introduces a broader class of varieties. We explain the connection of the latter with matrix Schubert varieties.

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1. Introduction

The aim of the present paper is to give a full account of the connection between certain geometric objects and quantum integrable models, following the ideas presented in [DFZJ06, KZJ07, DFZJ05b]. In sections 1.1–1.3 we recall the *Brauer loop scheme* we introduced in [KZJ07], and give a new interpretation of it in terms of the more common notion of *orbital varieties*. In section 1.4 we discuss the *quantum Knizhnik–Zamolodchikov equation*, and its relation to a refinement of the Brauer model studied in [DFZJ06]. We connect the two using polynomial-valued *multidegrees*, whose definition we recall in section 1.5. All these points are then developed in the rest of the paper.

1.1. **The Brauer loop scheme.**

Let $R_{\mathbb{Z}}$ denote the vector space of upper triangular complex matrices with rows and columns indexed by $\mathbb{Z}$. Despite the infinitude, any matrix entry in a product $A \cdot B$ is a sum of finitely many nonzero terms, so $R_{\mathbb{Z}}$ is an algebra. Let $R_{\mathbb{Z} \bmod N} \leq R_{\mathbb{Z}}$ denote the subalgebra of matrices with the periodicity

$$A_{ij} = A_{i+N,j+N} \quad \forall i, j \in \mathbb{Z};$$

a typical element looks like

$$
\begin{array}{ccccccccc}
\ddots & \ddots & \ddots & & & & \\
& R & L & Z & \dots & & \\
& & R & L & Z & \dots & \\
& & & R & L & Z & \dots \\
& & & & R & L & \dots \\
& & & & & R & \dots \\
& & & & & & \ddots
\end{array}
\qquad
\begin{array}{c}
R \in M_N(\mathbb{C}) \text{ upper triangular} \\
L, Z, \dots \in M_N(\mathbb{C})
\end{array}
$$

where R, on the main diagonal, is upper triangular. This subalgebra contains the "shift" matrix S carrying 1s just above the main diagonal, $S_{ij} = \delta_{i,j-1}$.

Any M in the quotient algebra

$$\mathcal{M}_N := R_{\mathbb{Z} \bmod N} / \langle S^N \rangle$$

is determined by the entries M_{ij} with $0 \leq i, j-i < N$, and this algebra is finite dimensional of dimension N^2. (In terms of the picture above, only R and the strict lower triangle of L remain well-defined in the quotient. We will use often this splitting of $\mathcal{M}_N$.) As explained in [KZJ07], this solvable algebra $\mathcal{M}_N$ is a degenerate limit of the usual matrix algebra $M_N(\mathbb{C})$.

Define the **Brauer loop scheme** $E \subseteq \mathcal{M}_N$ as the space of strictly upper triangular matrices M whose square is "zero", i.e. $M^2 \in \langle S^N \rangle$. We will occasionally find it useful to choose lifts $\widetilde{M} \in R_{\mathbb{Z} \bmod N}$ of its elements. We introduced this scheme in [KZJ07], under a different but equivalent definition.

This scheme E is reducible, and we call its top-dimensional components the **Brauer loop varieties**. They are naturally indexed by **link patterns** $\pi \in S_N$ (meaning, involutions with at most one fixed point), as we now recall from [KZJ07]. First, note that if we pick a representative $\widetilde{M} \in R_{\mathbb{Z} \bmod N}$ lying over some $M \in E$, then for any $i \in \mathbb{Z}/N\mathbb{Z}$,

$$
\begin{aligned}
(\widetilde{M}^2)_{i,i+N} &= \sum_j \widetilde{M}_{i,j}\, \widetilde{M}_{j,i+N} \\
&= \sum_{\substack{j \\ i<j<i+N}} M_{i,j}\, M_{j,i+N} \qquad \text{since } \widetilde{M} \text{ has zero diagonal,}
\end{aligned}
$$

so that $(\widetilde{M}^2)_{i,i+N}$ is well-defined (independent of the representative) even though $\widetilde{M}_{i,i+N}$ itself is not because of the $\langle S^N \rangle$ ambiguity. By abuse of notation, we shall simply denote it by $(M^2)_{i,i+N}$ in what follows.

Theorem 1 (Theorems 2 and 3 of [KZJ07]).

(1) *Let* $M \in E$.
Then there exists a link pattern π *such that* $(M^2)_{i,i+N} = (M^2)_{\pi(i),\pi(i)+N}$.

(2) *For general points of the Brauer loop scheme* E, *there are no additional equalities among the* $\{(M^2)_{i,i+N}\}$*; the link pattern* π *is unique.*

(3) *The map*

$$\{\textit{top-dimensional components of } E\} \longrightarrow \{\textit{link patterns}\},$$

which takes a component to the link pattern of a general point therein, is a bijection.

Brian Rothbach has recently shown that E is equidimensional [Ro10], which allows one to drop "top-dimensional" from the above statement.

To a link pattern π associate $\underline{\pi} \in E_\pi$ which is the matrix with 1's at entries $(i, \pi(i))$ with representatives mod N such that $i < \pi(i) < i + N$, and 0's elsewhere. We shall need the following (Theorem 5 of [KZJ07]).

Theorem 2. *The irreducible component E_π of E corresponding to the link pattern π satisfies the following equations:*

(1) $M^2 = 0$.
(2) $(M^2)_{i,i+N} = (M^2)_{\pi(i),\pi(i)+N}$.
(3) *For any matrix entry (i, j), $i < j < i + N$, we have $r_{ij}(M) \leq r_{ij}(\underline{\pi})$, where r_{ij} denotes the rank of the submatrix south-west of entry (i, j). In polynomial terms, this asserts the vanishing of all minors of size $r_{ij}(\underline{\pi}) + 1$ in the submatrix southwest of entry (i, j).*

In fact we conjectured that the equations of Thm. 2 define E_π.

1.2. **The orbital varieties of** $M^2 = 0$**.** Let $D \subseteq M_N(\mathbb{C})$ denote the closure of some conjugacy class of nilpotent matrices (though we will soon specialize to $D = \{M : M^2 = 0\}$), and let R_N denote the space of upper triangular matrices. In this generality, the intersection $D \cap R_N$ is called the **orbital scheme** of D, and its geometric components are called the **orbital varieties**. (Conventions differ about whether D should be a nilpotent orbit or its closure; these issues will not be relevant for us and we will always take D, and its orbital varieties, to be closed.) The orbital scheme carries an action by conjugation of B, the group of invertible upper triangular matrices; hence each orbital variety carries such an action too.

The orbital varieties were shown in [Sp76] to all have the same dimension $\frac{1}{2}\dim D$, and to be naturally indexed by standard Young tableaux, on the partition determined by the Jordan canonical form of generic elements of D.

In this paper we will only be interested in the case $D = \{M : M^2 = 0\}$ in even dimensions $N = 2n$. So the relevant partition is $(2, 2, \ldots, 2)$, and the standard Young tableaux correspond in a simple way with *noncrossing* chord diagrams. (We will correspond orbital varieties with noncrossing chord diagrams directly in section 2, and make no use of Young tableaux in this paper. There are a multitude of other interpretations of this Catalan number in [St97].)

The space of matrices is $(2n)^2$-dimensional, with $\dim D$ half that, and $\dim(D \cap R_N)$ half that again, so n^2. One reason this nilpotent orbit is easier to deal with than a general one is that it is **spherical**: it has only finitely many B-orbits. In particular, each component is a B-orbit closure. The set of orbits was described in [Me00]; we give a new way to index the orbits in section 2.

The Brauer loop scheme has two simple connections to the orbital scheme $D \cap R_N$, for which we will give deeper reasons in section 1.3. There is an inclusion

$$R_N \to \mathcal{M}_N, \qquad U \mapsto M \quad \text{where} \quad M_{ij} = \begin{cases} U_{ij} & 1 \leq i \leq j \leq N \\ 0 & 1 \leq i \leq N < j \end{cases}$$

that takes $D \cap R_N \hookrightarrow E$, and a projection

$$\mathcal{M}_N \to R_N, \qquad M \mapsto U \quad \text{where} \quad U_{ij} = M_{ij}, 1 \leq i \leq j \leq N$$

that takes $E \twoheadrightarrow D \cap R_N$. (In the (R, L) notation from section 1.1, the two maps are $R \mapsto (R, 0)$, $(R, L) \mapsto R$.) Moreover, the composite $D \cap R_N \hookrightarrow E \twoheadrightarrow D \cap R_N$ of these two maps is the identity. For each component E_π of E corresponding to a chord diagram π, one can compute the dimension of the projection to $D \cap R_N$; it turns out to be n^2 minus the number of crossings in π (Theorem 7).

1.3. **The normal cone to the orbital scheme.** The connection between the Brauer loop scheme and the orbital scheme is tighter than just indicated, as already implicit in [KZJ07], and as will be discussed in detail in section 3. We first recall the definition of the **normal cone** $C_X Y$ to a subscheme $X \subseteq Y$, both schemes affine. Say that $Y = \operatorname{Spec} A$ and X is cut out of Y by the vanishing of an ideal $I \leq A$. Then A is filtered by powers of the ideal, $A \geq I \geq I^2 \geq \ldots$, and $C_X Y$ is defined as the Spec of the associated graded algebra $\operatorname{gr} A := (A/I) \oplus (I/I^2) \oplus (I^2/I^3) \oplus \ldots$. Note that while there is no natural map $Y \twoheadrightarrow X$ reversing the inclusion $X \to Y$, there is a natural map $C_X Y \twoheadrightarrow X$ reversing a natural inclusion $X \hookrightarrow C_X Y$.

When X and Y are smooth, the projection $C_X Y \twoheadrightarrow X$ is a vector bundle, and the components of $C_X Y$ correspond $1:1$ to the components of X. More generally, if X°, Y° denote the smooth loci, then $C_X Y$ contains $C_{X^\circ} Y^\circ$ as an open subset, and the components of $C_{X^\circ} Y^\circ$ correspond 1:1 to the (generically reduced geometric) components of X. However, $C_{X^\circ} Y^\circ$ may miss some of the components of $C_X Y$, and one can take this as a measure of the nonsmoothness of the embedding $X \hookrightarrow Y$.

Though we won't use it, we suggest a physical description (which we learned in a lecture of Ed Witten) to help understand the role of the "extra components" of the normal cone. Imagine that Y is a phase space carrying a Hamiltonian $H + h$, where $H \geq 0$ grows very quickly and h does not. Then a particle traveling on Y is likely to stay near the $H = 0$ locus, which we call X. So at low energies, one can generally (really, on X°) treat the system as having the phase space X with Hamiltonian h, plus small oscillations in the normal directions to X°. However, where H vanishes to high order, one can get further off X without high energies, and detect more of the geometry of Y; this is exactly where X is singular. Having "extra" components of the normal cone is a sign that (X, h) is not a good approximation to $(Y, H + h)$, even at low energies, near the projections of these extra components to X.

When X, Y are each equidimensional, one can measure "how extra" a component of $C_X Y$ is. Since the projection $C_X Y \twoheadrightarrow X$ is surjective, each component of X is the image of some component of $C_X Y$, but not every component of $C_X Y$ (which, like Y, is also equidimensional) projects onto a component of X; it may project to something lower-dimensional. So to each component of $C_X Y$ we can associate the codimension inside X of the projection of the component.

In the case at hand, Y is the nilpotent orbit and X is the orbital scheme. We conjectured in [KZJ07, Theorem 10] (in slightly different language) that in the right coordinates,

$$C_{D \cap R_N} D = \{M \in \mathcal{M}_N : M^2 = 0\};$$

we were able to prove the $\subseteq$ inclusion, and that these two schemes agree in top dimension. (The scheme E is smaller than these, because in its definition we impose that M has diagonal entries *equal* to zero, rather than just squaring to zero. Of course this makes no difference on the level of varieties.)

Here $D \cap R_N$ has many fewer components than $C_{D\cap R_N} D$ (they correspond to noncrossing chord diagrams, rather than all chord diagrams). In Theorem 7 we show that the codimension inside X of the projection of a component is exactly the number of crossings in the corresponding chord diagram.

1.4. **Integrability and the qKZ equation.** In [dGN05], a certain Markov process on the set of (crossing) link patterns was considered. The motivation was that the Markov matrix is actually a *quantum integrable* transfer matrix related to the Brauer algebra with parameter β, at the special value $\beta = 1$ of the parameter. A remarkable conjecture of [dGN05] (now a theorem) is that certain components of the equilibrium distribution eigenvector of this Markov process can be identified, after dividing them by the smallest component, with degrees of certain algebraic varieties.

The model was further studied in [DFZJ06], where several important properties were shown. First, the model can naturally be made inhomogeneous, and the introduction of the inhomogeneities z_i make these equilibrium probabilities polynomials (again, up to normalization) in the variables z_i. On the geometric side, this correponds to generalizing degrees to *multidegrees*, that is, enlarging the torus action, as will be explained in the next section. Secondly, the main method used in [DFZJ06], inherited from [DFZJ05a], is to write certain "exchange relations": these express the effect of interchange of variables z_i, z_{i+1} as a linear operator acting on the equilibrium distribution vector. The exchange relations appear in multiple contexts in the study of quantum integrable models, but in particular, supplemented with an appropriate cyclicity property, they are related to the so-called quantum Knizhnik–Zamolodchikov (qKZ) equation [Sm86, FR92].

More progress was made in [KZJ07], where *every* component of the Markov process eigenvector was given a geometric meaning: these are the (multi)degrees of the Brauer loop varieties. The central role of the exchange relation, already pointed out in [DFZJ06], is developed further in [KZJ07]. The present work will complete (in section 4) the general program outlined in these two prior papers: the exchange relation will be entirely explained geometrically as the translation at the level of equivariant cohomology of certain elementary geometric operations on the irreducible components of the Brauer loop scheme.

Note that even though there exists a solution of the Yang–Baxter equation for arbitrary values of the parameter β of the Brauer algebra, one cannot define a corresponding Markov process, as was the case at $\beta = 1$. One can however introduce a qKZ equation and try to look for certain polynomial solutions which would generalize the equilibrium vector of the Markov process at $\beta = 1$. This provides a much more natural framework to study the Brauer loop scheme, and is what is considered in the present work. The parameter β can be thought as yet another enlargement of the torus action (the additional circle action was in fact mentioned in the last section of [KZJ07]).

1.5. **Torus actions and multidegrees.** Let $T \cong (\mathbb{C}^\times)^k$ be a complex torus, and consider the class of linear T-representations W containing T-invariant closed subschemes $X \subseteq W$. To each such pair $X \subseteq W$ we will assign a polynomial $\mathrm{mdeg}_W X \in \mathrm{Sym}(T^*) \cong \mathbb{Z}[z_1, \ldots, z_k]$ called the **multidegree** of X. (Here $\mathrm{Sym}(T^*)$ denotes the symmetric algebra on the lattice T^* of characters of T.) Our reference for multidegrees is [MS04].

This assignment can be computed using the following properties (as in [Jo97]):

1. If $X = W = \{0\}$, then $\operatorname{mdeg}_W X = 1$.
2. If the scheme X has top-dimensional components X_i, where $m_i > 0$ denotes the multiplicity of X_i in X, then $\operatorname{mdeg}_W X = \sum_i m_i \operatorname{mdeg}_W X_i$. This lets one reduce from the case of schemes to the case of varieties (reduced irreducible schemes).
3. Assume X is a variety, and H is a T-invariant hyperplane in W.
   (a) If $X \not\subset H$, then $\operatorname{mdeg}_W X = \operatorname{mdeg}_H (X \cap H)$.
   (b) If $X \subset H$, then $\operatorname{mdeg}_W X = (\operatorname{mdeg}_H X) \cdot$ (the weight of T on W/H).
   (c) Combining (a) and (b) when $X \not\subset H$, we have $\operatorname{mdeg}_W (X \cap H) = (\operatorname{mdeg}_W X)\,(\operatorname{mdeg}_W H)$. We ask [1] this to hold even when $H \subseteq W$ is just a T-invariant hyper*surface*.

One can readily see from these properties that $\operatorname{mdeg}_W X$ is homogeneous of degree $\operatorname{codim}_W X$, and is a positive sum of products of the weights of T on W. We explore this further in Lemma 12.

The varieties for which we will need the multidegrees are the orbital varieties and the Brauer loop varieties. In both cases, W will be a space of zero-diagonal matrices, and the $(N+2)$-dimensional torus T that acts on it will have three parts: two dimensions by scaling certain halves of the matrices, and the other N by conjugating by the invertible diagonal matrices in $R_{\mathbb{Z} \bmod N}$. We will denote by $\{A, B, z_1, \ldots, z_N\}$ the obvious basis of the weight lattice T^*.

For orbital varieties, the vector space W will be $\mathfrak{n}$, the space of strictly upper triangular matrices. The first circle in T acts by global rescaling, and the second acts trivially, so the T-weights are $A + z_i - z_j$, $i < j \in 1, \ldots, N$. For each noncrossing chord diagram π, let

$$J_\pi := \operatorname{mdeg}_{\mathfrak{n}} \mathcal{O}_\pi,$$

which is called the **(extended) Joseph polynomial** [Jo84] of the orbital variety. (Indeed, Joseph invented multidegrees for exactly this application.) The "extended" refers to the fact that Joseph did not consider the scaling action; his polynomials correspond to the specialization $A = 0$.

For Brauer loop varieties, the vector space W will be $\mathcal{M}_N^{\Delta=0}$, where the $\Delta = 0$ indicates the subspace of matrices with zero diagonal. Separate the matrix entries into the "R" group, those matrix entries M_{ij} with $\lfloor i/N \rfloor = \lfloor j/N \rfloor$, and the "L" group, which are the rest. (This matches the picture in section 1.1.) The first circle acts by scaling R, and trivially on L, the second circle acts trivially on R, and by scaling on L. The weights on $\mathcal{M}_N$ are $A + z_i - z_j$, $B + z_j - z_i$, $i < j = 1, \ldots, N$. For each chord diagram π, let

$$\Psi_\pi := \operatorname{mdeg}_{\mathcal{M}_N^{\Delta=0}} E_\pi$$

be the **Brauer loop polynomial**. (In [KZJ07], we did not consider until section 8 the separate action on the R and L parts, and before that recovered only the specialization $A = B$ of the polynomials presented here.)

An interesting, explicitly cyclic-invariant, reformulation of the weights is obtained by introducing a redundant set of variables z_i, $i \in \mathbb{Z}$, with the relations $z_{i+N} = z_i + A - B$. Then the weight of M_{ij} is simply $A + z_i - z_j$ for any i, j. This notation will be used in what follows.

Note that the maps in section 1.2 are T-equivariant, which in Theorem 7 will help us relate Joseph and Brauer loop polynomials.

[1] This actually follows from the other axioms, since they imply that $\operatorname{mdeg}_W X$ can be computed from the multigraded Hilbert series of X, and it is easy to relate the Hilbert series of X and $X \cap H$.

These multidegrees will be the main concern of section 5.

2. The orbital varieties of $\{M^2 = 0\}$

2.1. **The poset of B-orbits of** $\{M^2 = 0\}$**.** Recall that D denotes the nilpotent orbit closure $D := \{M : M^2 = 0\}$ inside $\mathfrak{gl}_N$ (in this section the parity condition on N is relaxed unless stated otherwise). This nilpotent orbit is much easier to study than a general one, in that it is **spherical**: the Borel subgroup B acts on D with finitely many orbits. (Moreover, the other nilpotent orbits of $\mathfrak{gl}_N$ with this property, such as $\{\vec{0}\}$, are all contained in D.) This finite set of orbits naturally forms a ranked poset, with the rank given by the dimension of the orbit, and the partial order by inclusion of orbit closures. A full description of this ranked poset appears in [Ro].

We will not study all the B-orbits on D, but focus on the B-invariant orbital scheme $D \cap R_N$, whose corresponding subposet was determined in [Me00, Me06]. Each component of $D \cap R_N$ (i.e. each orbital variety) itself has finitely many B-orbits, and in particular is the closure of a B-orbit, corresponding to a maximal element of the poset.

Theorem 3. [Me00] *For each orbit* $\mathcal{O}$ *of* B *on* $D \cap R_N$*, there exists a unique involution* $\pi \in S_N$ *such that* $\mathcal{O} = B \cdot \pi_<$.

We will draw these involutions as chord diagrams on the interval, with an arch connecting $i \leftrightarrow \pi(i)$ for $i \neq \pi(i)$, and a vertical half-line from i for each fixed point $i = \pi(i)$, drawn so that any two **curves** (meaning, arch or half-line) cross transversely and at most once. This encoding will make it easy to describe the dimension of the orbit and the covering relations in the poset. While these were already computed in [Me00, Me06], our description is sufficiently different that we find it simpler to give independent proofs.

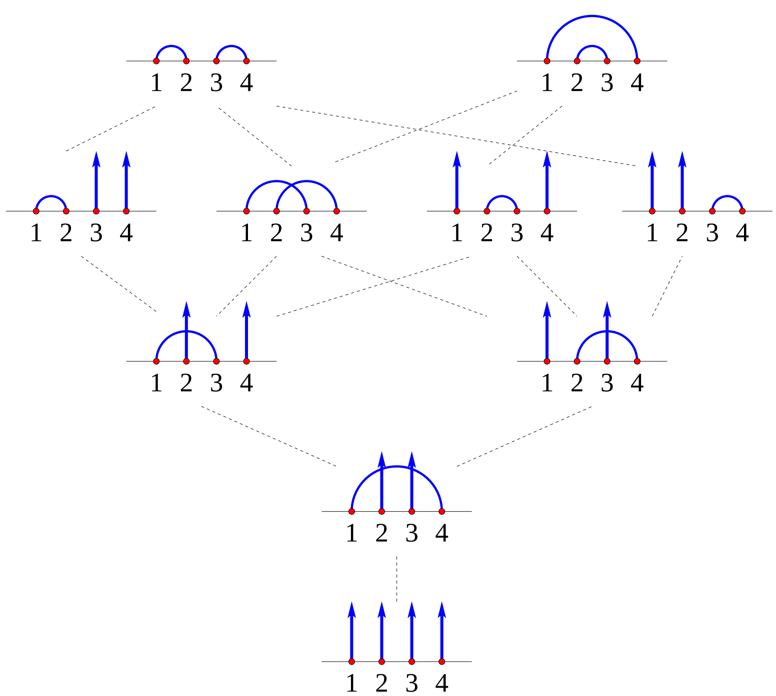


Figure 1. The poset of B-orbits for $N = 4$. The row gives the dimension, computable from Theorem 4, from 4 at the top down to 0 for the orbit $\{0\}$.

Theorem 4. *Let $\pi \in S_N$ be an involution, drawn as a chord diagram on the interval. Then the corresponding* B*-orbit has dimension*

$$\dim B \cdot \pi_< = \#\textit{arches} + \#\textit{noncrossing pairs (arch, arch or half-line)}$$
$$= \#\textit{arches} \cdot (\#\textit{arches} + \#\textit{half-lines}) - \#\textit{crossings}.$$

The maximum dimension $\lfloor N^2/4 \rfloor$ is achieved iff π has no crossings and at most one half-line, iff $\overline{B \cdot \pi_<}$ is an orbital variety of D.

More generally, if we require that π has at most k 2-cycles, then the maximum dimension $k(N-k)$ is achieved iff there are indeed k 2-cycles and no crossings, iff $\overline{B \cdot \pi_<}$ is an orbital variety for the nilpotent orbit closure $\mathcal{O} = \{M : M^2 = 0, \operatorname{rank} M \leq k\}$. This latter statement – that the components of $\mathcal{O} \cap R_N$ are all the same dimension – holds for any nilpotent orbit $\mathcal{O}$ [Sp76]; in fact the dimension is $\frac{1}{2}\dim \mathcal{O}$.

For an involution π, write

$$\operatorname{rank} \pi_{[ij]} := \#\{i \leq a < b \leq j : \pi(a) = b\},$$

i.e. the number of complete arches sitting between positions i and j of the chord diagram.

Theorem 5. [Me06, Ro] *For two involutions $\pi, \rho \in S_n$,*

$$\overline{B \cdot \pi_<} \supseteq B \cdot \rho_< \quad \iff \quad \operatorname{rank} \pi_{[ij]} \geq \operatorname{rank} \rho_{[ij]} \quad \forall i < j.$$

Proof of $\Longrightarrow$. Let $M_{\geq i, \leq j}$ denote the part of M southwest of (i, j), i.e.

$$(M_{\geq i,\leq j})_{kl} = \begin{cases} M_{kl} & \text{if } k \geq i \text{ and } l \leq j \\ 0 & \text{otherwise.} \end{cases}$$

In the case $M = \pi_<$, we have $\operatorname{rank} (\pi_<)_{\geq i,\leq j} = \operatorname{rank} \pi_{[ij]}$.

The conjugation action of B on M restricts, in the following sense, to an action on each southwest part:

$$(b \cdot M)_{\geq i,\leq j} = (b \cdot M_{\geq i,\leq j})_{\geq i,\leq j}.$$

This implies that $\operatorname{rank} M_{\geq i,\leq j}$ is invariant under B-conjugation.

Consequently, if $M' \in B \cdot M$, then $\operatorname{rank} M'_{\geq i,\leq j} = \operatorname{rank} M_{\geq i,\leq j}$ for all i, j. The semicontinuity of rank gives us an inequality on the closure:

$$M' \in \overline{B \cdot M} \quad \Longrightarrow \quad \operatorname{rank} M_{\geq i,\leq j} \geq \operatorname{rank} M'_{\geq i,\leq j} \quad \forall i, j.$$

Now apply this to $M' = \rho$, $M = \pi$ to obtain the desired statement. □

We will prove Theorem 4 and the other half of Theorem 5 by an analysis of the covering relations in the poset of orbit closures (the "moves" from [Ro]). We encourage the reader to reconstruct the poset in Figure 1 from the top down using the following Proposition.

Proposition 1. *Let $\pi \in S_N$ be an involution, with an associated chord diagram also called π. Construct a new chord diagram ρ in one of three ways:*

1. *If two arches in π border a common region, but don't cross, make them touch and turn that into a new crossing;*
2. *if an arch and a half-line in π border a common region, but don't cross, make them touch and turn that into a new crossing;*

(3) *if an arch crosses all the half-lines, and borders the unbounded region, break it there into two half-lines.*

Then $\pi > \rho$ *in the poset of orbit closures, i.e.* $\overline{B \cdot \pi_<} \supset B \cdot \rho_<$. *(We will later prove these to be covering relations, and all of them.)*

Proof. In each of these cases, we will construct a one-parameter family of group elements $b(t) \in B$ such that $\lim_{t\to 0} b(t) \cdot \pi = \rho$, where $\cdot$ denotes the conjugation action. These π and ρ only differ in a few columns and rows, and we will be able to take $b(t)$ the identity outside those, making it possible to write down $b(t)$ in a small space.

1. If one arch contains the other, the relevant submatrices are

$$\begin{pmatrix} 1 & 1 & & \\ & t & & \\ & & 1 & -1 \\ & & & -t \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ & 0 & 1 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & t & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \dashrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \text{ as } t \to 0.$$

If they are side by side:

$$\begin{pmatrix} t & & & \\ & 1 & 1 & \\ & & -t & \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & t & 1 & 0 \\ & 0 & 0 & 1 \\ & & 0 & t \\ & & & 0 \end{pmatrix} \dashrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \text{ as } t \to 0.$$

2. Assume (by symmetry) that the half-line is left of the arch:

$$\begin{pmatrix} 1 & 1 & \\ & t & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & t \\ & & 0 \end{pmatrix} \dashrightarrow \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \text{ as } t \to 0.$$

3. We won't here need to use the condition that the arch crosses all half-lines – it is only included to later ensure that this is a covering relation.

$$\begin{pmatrix} t & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} = \begin{pmatrix} 0 & t \\ & 0 \end{pmatrix} \dashrightarrow \begin{pmatrix} 0 & 0 \\ & 0 \end{pmatrix} \text{ as } t \to 0.$$

□

When looking for the B-orbits covered in this poset by a given B-orbit $B \cdot \pi_<$, one must be careful to consider *all* the ways to draw the chord diagram of π, as different drawings may make different pairs of chords adjacent. An example is in Figure 2.

Proof of Theorem 4. Let d_π denote the statistic in Theorem 4; we wish to prove $d_\pi = \dim B \cdot \pi_<$. One can do this (as in [Me00]) by computing the commutant in B of $\pi_<$, but we will find it instructive to use Proposition 1.

First we prove that for each of the moves in Proposition 1, $d_\pi = d_\rho + 1$. For the first two constructions it is essentially obvious – the number of arches and curves don't change, and exactly one crossing is created.

For the third move, let a be the number of arches, $N - 2a$ the number of half-lines, and c the number of crossings, so $d_\pi = a(N - a) - c$. When we break the arch, we lose $N - 2a$ crossings of it by half-lines, so $d_\rho = (a-1)(N-(a-1)) - (c - (N-2a)) = d_\pi - 1$ as desired.

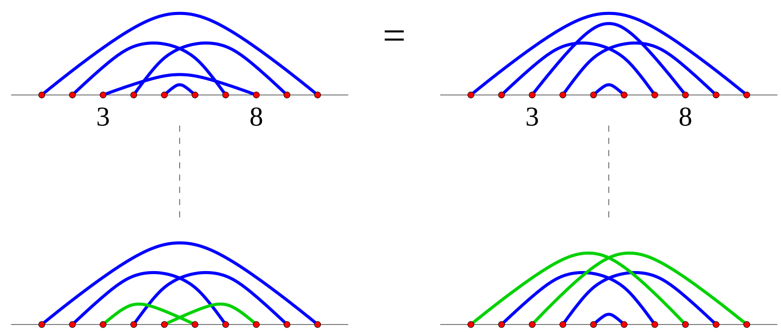


FIGURE 2. The top two chord diagrams are of the same involution π, with the $3 \leftrightarrow 8$ chord placed differently. In either diagram, the $3 \leftrightarrow 8$ chord is brought into contact with another chord, leading to the lower two diagrams. In those two we emphasize the newly crossing chords.

We now embed π in a maximal chain in the poset, first building upward (using the moves in reverse). Alternate between two strategies: replace every crossing with)(, then when there are no crossings, join the leftmost half-line to the right-most, making an arch that crosses the remaining half-lines and doesn't cross any other arch twice (or indeed at all). This process stops at a noncrossing chord diagram $\pi^+ \geq \pi$ with at most one half-line.

To go downward in the poset, look first for a half-line at position i and an arch $j \leftrightarrow k$ not crossing it. By reflecting, we can assume for discussion that $i < j < k$. Find such an arch with smallest j. If there is a half-line at $j-1$ (e.g. if $i = j-1$), then we can apply move #2 to that half-line and the $j \leftrightarrow k$ arch and thus go downward in the poset. If there is an arch at $j-1$ and $\pi(j-1) < j-1$, then we can apply move #1 to make the arches $\pi(j-1) \leftrightarrow j-1$ and $j \leftrightarrow k$ cross. Otherwise there is an arch at $j-1$ with $\pi(j-1) > j-1$, hence not crossing the half-line at i, contradicting our choice of smallest j.

Once that is done, every half-line crosses every arch. Now if there are any arches left, we can apply move #3 to any arch touching the unbounded region. (Then return to the paragraph above.) This process stops only when there are no arches, i.e. at the identity permutation $\mathbf{1}$.

We have thus constructed a sequence $(\pi^+, \ldots, \pi, \ldots, \mathbf{1})$, each related by a move from Proposition 1, with $(d_{\pi^+} = \lfloor N^2/4 \rfloor, \ldots, d_\pi, \ldots, d_{\mathbf{1}} = 0)$ therefore decreasing by 1 at each step. Correspondingly, we can compute the sequence $(\dim B \cdot \pi^+_<, \ldots, \dim B \cdot \pi_<, \ldots, 0)$ of dimensions of the orbits, which must strictly decrease at each step (since for any variety, $\dim(\overline{X} \setminus X) < \dim X$). All that remains is to know that $\dim(D \cap R_n) = \frac{1}{2} \dim D$ [Sp76], straightforwardly computed to be $\lfloor N^2/4 \rfloor$, and this is an upper bound on $\dim B \cdot \pi^+_<$. $\square$

In particular, this computation of the rank function proves that the moves in Proposition 1 are indeed covering relations.

Proof of Theorem 5, $\Longleftarrow$. We first analyze the effect of the moves $\pi \mapsto \rho$ from Proposition 1 on the rank function $\operatorname{rank} \pi_{[ij]}$, leaving the details of the verification to the reader.

For moves 1 and 2, $\operatorname{rank} \rho_{[ij]} = \operatorname{rank} \pi_{[ij]}$ unless the interval $[ij]$ fully contains one arch and does not contain any end of the other curve, in which case $\operatorname{rank} \rho_{[ij]} = \operatorname{rank} \pi_{[ij]} - 1$. For move 3, $\operatorname{rank} \rho_{[ij]} = \operatorname{rank} \pi_{[ij]}$ unless the interval $[ij]$ fully contains the arch, in which case $\operatorname{rank} \rho_{[ij]} = \operatorname{rank} \pi_{[ij]} - 1$.

In particular, $\operatorname{rank} \rho_{[ij]} \leq \operatorname{rank} \pi_{[ij]}$ for all $i < j$, and for some $[ij]$ the inequality is strict, making inductive arguments possible.

Now let $\pi \neq \rho$ be chord diagrams on the interval (possibly with half-lines) such that $\operatorname{rank} \pi_{[ij]} \geq \operatorname{rank} \rho_{[ij]} \quad \forall i < j$. We wish to show that there exists a move $\pi \mapsto \pi'$ such that we again have $\operatorname{rank} \pi'_{[ij]} \geq \operatorname{rank} \rho_{[ij]} \quad \forall i < j$. Then by Proposition 1 and induction, we can infer that $\overline{B \cdot \pi_<} \supset B \cdot \rho_<$.

Here it is useful to distinguish move (1a) where the two arches to be crossed are next to each other and move (1b) where the two arches are nested one inside the other.

Let $b < c$ be such that $\operatorname{rank} \pi_{[bc]} > \operatorname{rank} \rho_{[bc]}$ with $c - b$ minimized. Then π necessarily has an arch connecting b and c. Next consider any pair $a \geq 0$, $d \leq N + 1$ such that $\operatorname{rank} \pi_{[ij]} > \operatorname{rank} \rho_{[ij]}$ for all $a < i \leq b$ and $c \leq j < d$, and such that a is minimum, d is maximum, for that condition. There are four cases:

- π possesses an arch inside the square $[a, b[\times]c, d]$, i.e. $c < \pi(i) \leq d$ for some $a \leq i < b$. Pick i such that $\pi(i) - i$ minimal, and apply covering relation (1b) to arches b, c and $i, \pi(i)$.

  In all other cases we assume that π has no arch inside $[a, b[\times]c, d]$ (i.e. no arch "covers" (b, c)). Note that this implies $\operatorname{rank} \pi_{[ij]} = \operatorname{rank} \rho_{[ij]} + 1$ for $a < i \leq b$, $c \leq j < d$, and furthermore that one cannot have simultaneously $a > 0$ and $d < N + 1$.
- $a = 0$ and $d = N + 1$. Then arch b, c is connected to the outside. If there exists a half-line outside arch b, c, then apply covering relation (2) to b, c and the closest half-line. If not, apply covering relation (3) to b, c.
- $a = 0$ and $d < N + 1$. If there exists a half-line between c and d (including d), apply covering relation (2) to the arch (b, c) and the closest half-line. If not apply covering relation (1a) to (b, c) and $(d, \pi(d))$.
- $a > 0$ and $d = N + 1$. If there exists a half-line between b and a (including a), apply covering relation (2) to the arch (b, c) and the closest half-line. If not apply covering relation (1a) to (b, c) and $(\pi(a), a)$.

□

Corollary (of proof). *The covering relations in Proposition 1 are all the covering relations.*

2.2. **Joseph–Melnikov polynomials.** Let $\pi \in S_N$ be an involution, and $\overline{B \cdot \pi_<}$ the corresponding orbit closure in the orbital scheme of $\{M^2 = 0\}$. Then we define the **Joseph–Melnikov polynomial** J_π to be the multidegree of $\overline{B \cdot \pi_<}$ exactly as we did the (extended) Joseph polynomials, i.e. inside the strictly upper triangular matrices, with respect to the action of the scaling circle and the diagonal matrices. In particular, if π has no crossings, then the orbit closure is an orbital variety, and the Joseph–Melnikov polynomial is its Joseph polynomial.

Though we won't make direct use of it, we point out one case of particular interest. Let $\rho \in S_n$ be an arbitrary permutation, and associate an involution π of $1, \ldots, N = 2n$ as follows:

$$\pi(i) = \begin{cases} n + \rho(n + 1 - i) & i = 1, \ldots, n \\ n + 1 - \rho^{-1}(i - n) & i = n + 1, \ldots, N \end{cases}$$

Such involutions were already considered in section 5 of [KZJ07], forming the so-called *permutation sector*. Then $\pi_<$ is not only upper triangular, but supported in the upper right quarter of this $2n \times 2n$ matrix, where it matches the permutation matrix of ρ (except for being upside down). The action of $B_N \subseteq M_N(\mathbb{C})$ by conjugation splits into separate actions of $B_n \subseteq M_n(\mathbb{C})$ by left and right multiplication, and $\overline{B \cdot \pi_<}$ is linearly isomorphic to a *matrix Schubert variety*. Our reference for these varieties is [MS04].

Proposition 2. *Let $\rho \in S_n$, and construct $\pi \in S_N$ as above. Then the Joseph–Melnikov polynomial J_π is essentially the double Schubert polynomial S_ρ of ρ:*

$$J_\pi = S_\rho(A + z_n, \dots, A + z_1;\ z_{n+1}, \dots, z_{2n}) \prod_{1 \le i < j \le n} (A + z_i - z_j) \prod_{n+1 \le i < j \le 2n} (A + z_i - z_j)$$

Proof. One definition of the double Schubert polynomial is as the multidegree of $\overline{B_- \rho B} \subseteq M_n(\mathbb{C})$, where B_- denotes the lower triangular matrices, with respect to the $2n$-dimensional torus formed from the diagonal matrices in B_- and B. One could separately include the action of the scaling circle, too, but this is not traditional, since the scalar matrices in B (or in B_-) already give the scaling action.

To compare these two multidegrees, we must first relate the two ambient spaces. The Joseph–Melnikov polynomial is defined using the space of strictly upper triangular $2n \times 2n$ matrices, which we identify with the space of $n \times n$ matrices (thought of as the upper right corner) times two triangles' worth of matrix entries. Those triangles account for the latter two factors in the formula. The action by conjugation, when restricted to these upper-right square matrices, becomes left and right action by upper triangular matrices: therefore one must additionally reverse the indices $i \mapsto n + 1 - i$ of the rows to recover matrix Schubert varieties.

To compare the action of the torus in $M_N(\mathbb{C})$ plus the scaling circle to the action of the diagonal matrices in B_- and B, it is easiest to look at the weight of the (i, j) entry on $M_n(\mathbb{C})$: for the first torus the weight is $A + z_i - z_j$, for the second the weight is $x_{n+1-i} - y_j$ (with respect to the usual notation, in which the multidegree is $S_\rho(x_1, \dots, x_n;\ y_1, \dots, y_n)$). This suggests the variable substitution we used in the given formula.

(This substitution is not unique; for example, $S_\rho(z_n, \dots, z_1;\ z_{n+1} - A, \dots, z_{2n} - A)$ would work equally well. The nonuniqueness can be traced to the fact that the torus in $M_N(\mathbb{C})$ does not act faithfully on the subspace containing $B \cdot \pi_<$.) $\square$

Recall that multidegrees have an automatic positivity property [MS04]: the multidegree of $X \subseteq V$ is a positive sum of products of T-weights from V (with no more repetition of weights than occurs in V). One reason to move beyond Joseph polynomials to the larger family of Joseph–Melnikov polynomials is to give an inductive formula for them that is *manifestly positive* in this sense. The following is adapted from [Ro], which deals with the subtler case of arbitrary B-orbits in $\{M^2 = 0\}$.

Theorem 6. [Ro] *Let π be an involution, and $a < b$ a minimal chord in π, i.e. $\pi(a) = b$ and $\nexists c, d$ with $a < c < d < b$, $\pi(c) = d$.*

Let ρ vary over the set of involutions such that π covers ρ in the poset of B-orbits, and there is no chord connecting a, b. Then for each such ρ we have $A + z_a - z_b \mid J_\rho$, and

$$J_\pi = \sum_\rho \frac{J_\rho}{A + z_a - z_b}.$$

Proof sketch. Part of this is quite direct from the properties we used to define multidegrees. We slice $\overline{B \cdot \pi_<}$ with the hyperplane $\{M_{ab} = 0\}$, which does not contain it since $\pi(a) = b$. By the other condition on a, b, the intersection is again B-invariant.

Hence the intersection is supported on $\bigcup_\rho \overline{B \cdot \rho}$, and it remains to check that the multiplicities are all 1, a tangent space calculation done in [Ro].

(It is interesting to note that the same construction, applied to more general B-orbit closures in $\{M^2 = 0\}$, can produce multiplicities 1 or 2.) ☐

Combining Theorem 6 with Proposition 2, one obtains an inductive positive formula for double Schubert polynomials. This turns out to be exactly the "transition formula" of Lascoux [La00].

Example. Consider the case of maximal rank in size $N = 2n = 4$. There are 3 involutions of $\{1, 2, 3, 4\}$ without fixed points, which we denote according to their cycles: $(12)(34)$, $(14)(23)$, $(13)(24)$. The first two are noncrossing, while the third one has one crossing. The corresponding Joseph–Melnikov polynomials are

$$\begin{aligned}
J_{(12)(34)} &= (A + z_2 - z_3)(2A + z_1 - z_4) & M_{23} = 0 \quad (M^2)_{14} = 0 \\
J_{(14)(23)} &= (A + z_1 - z_2)(A + z_3 - z_4) & M_{12} = M_{34} = 0 \\
J_{(13)(24)} &= (A + z_1 - z_2)(A + z_2 - z_3)(A + z_3 - z_4) & M_{12} = M_{23} = M_{34} = 0.
\end{aligned}$$

(These subvarieties are complete intersections, using the indicated equations on $\{M_{ij}\}$, which explains why they factor.) The first two correspond to orbital varieties, whereas the third corresponds to a higher codimension B-orbit. The last two form the "permutation sector", that is, once divided by their common factor $(A + z_1 - z_2)(A + z_3 - z_4)$, we recover the specializations of the double Schubert polynomials $S_1 = 1$ and $S_2(A{+}z_2, A{+}z_1; z_3, z_4) = A + z_2 - z_3$.

Remark: Since finishing this section, we received the preprint [Me] with which it has some overlap.

3. Specialization of multidegrees and normal cones

In the first subsection we prove some general results about multidegrees in normal cones, without explicit reference to the Brauer loop scheme or orbital varieties. In the second we give a divisibility/vanishing condition on a multidegree. In the third subsection we use these results to relate Brauer loop polynomials and Joseph–Melnikov polynomials, and to prove geometrically some algebraic results from [DFZJ06].

3.1. **The leading form of a multidegree.**

Fix two finite-dimensional complex vector spaces V, W, and let $\mathbb{C}^\times$ act on $V \oplus W$ with weight 0 on V and weight 1 on W. Since we will be interested in it geometrically rather than linearly, we will usually denote this space $V \times W$.

Lemma 1. *Let $X \subseteq V \times W$ be a closed $\mathbb{C}^\times$-invariant subscheme of $V \times W$. Then the projection of X to V is $X \cap V$ (and is, in particular, closed).*

Proof. Let $\pi : V \times W \to V$ denote the projection, which acts as the identity on V. Then $\pi(X) \supseteq \pi(X \cap V) = X \cap V$, proving one inclusion.

For the opposite inclusion, let $(\vec{v}, \vec{w}) \in X$, with $\pi(\vec{v}, \vec{w}) = (\vec{v}, \vec{0})$. Then by the invariance, $t \cdot (\vec{v}, \vec{w}) \in X$ for all $t \in \mathbb{C}^\times$, and $t \cdot (\vec{v}, \vec{w}) = (\vec{v}, t\vec{w})$.

By the assumption that X is closed, we know that the limit $\lim_{t \to 0}(\vec{v}, t\vec{w}) \in X$. That limit is $(\vec{v}, \vec{0})$. This proves the opposite inclusion. $\square$

For an example where the conclusions fail, consider the hyperbola $vw = 1$ in the plane, whose intersection with $w = 0$ is empty, and whose projection to the V-axis hits everything but 0.

There is a normal cone implicit in the conditions in lemma 1, in that $X \cong C_X(X \cap V)$. However, we will stick to the language used in that lemma for the rest of the section.

In addition, in the rest of this section V, W will carry actions of a torus T, making $V \oplus W$ a representation of $T \times \mathbb{C}^\times$. Let B be a generator of the weight lattice of $\mathbb{C}^\times$. Then multidegrees of $(T \times \mathbb{C}^\times)$-invariant subschemes of $V \times W$ are elements of the polynomial ring $(\mathrm{Sym}T^*)[\mathrm{B}]$.

In the following Proposition, we will often want to pick out those terms of an element of $(\mathrm{Sym}T^*)[\mathrm{B}]$ carrying the leading power of B. Denote this operator by init_B, for example $\mathrm{init}_\mathrm{B}(\mathrm{B}x + \mathrm{B}^2 y + \mathrm{B}^2 z) = \mathrm{B}^2(y + z)$, and call $\mathrm{init}_\mathrm{B} p$ the **B-leading form of** p.

Proposition 3. *Let $X \subseteq V \times W$ be a T-invariant irreducible subvariety of a direct sum of T-representations, and Π the projection of X to V. (Hence Π is closed by Lemma 1, and irreducible.) Let $\mathrm{init}_\mathrm{B}\mathrm{mdeg}_{V\times W} X = \mathrm{B}^p m$, $m \in \mathrm{Sym}T^*$.*

Then $p = \dim W + \dim \Pi - \dim X$, and m is a positive integer multiple of $\mathrm{mdeg}_V \Pi$.

Proof. The proof is by induction on $\dim W$. The base case $\dim W = 0$ is easy enough: $X = \Pi$, $p = 0$, and $m_p = 1\mathrm{mdeg}_V \Pi$.

For $\dim W > 0$, let H be a T-invariant hyperplane in W, with λ the T-weight on W/H. There are two cases to consider: $X \subseteq H$ (easy) or $X \not\subseteq H$ (harder).

If $X \subseteq H$, then $\mathrm{mdeg}_{V\times W} X = (\mathrm{B}+\lambda)\mathrm{mdeg}_{V\times H} X$, hence $\mathrm{init}_\mathrm{B}\mathrm{mdeg}_{V\times W} X = \mathrm{B}\,\mathrm{init}_\mathrm{B}\mathrm{mdeg}_{V\times H} X$. By the inductive hypothesis, $\mathrm{init}_\mathrm{B}\mathrm{mdeg}_{V\times H} X$ is a positive multiple of $\mathrm{B}^{\dim H + \dim \Pi - \dim X}\mathrm{mdeg}_V \Pi = \mathrm{B}^{p-1}\mathrm{mdeg}_V \Pi$. Hence $\mathrm{B}\mathrm{init}_\mathrm{B}\mathrm{mdeg}_{V\times H} X$ is $\mathrm{B} \cdot \mathrm{B}^{p-1}\mathrm{mdeg}_V \Pi = \mathrm{B}^p\mathrm{mdeg}_V \Pi$ as claimed.

If $X \not\subseteq H$, then $\mathrm{mdeg}_{V\times W} X = \mathrm{mdeg}_{V\times H}(X \cap H)$. Let $\{X_i\}$ be the components of $X \cap H$ of dimension $\dim X - 1$, appearing with multiplicities m_i, so $\mathrm{mdeg}_{V\times H}(X \cap H) = \sum_i m_i \mathrm{mdeg}_{V\times H} X_i$. Note that $\dim W - \dim X = \dim H - \dim X_i$.

By Lemma 1, the projection of $X \cap H$ to V is Π. Hence the projection of any X_i to V, call it Π_i, is contained in Π. Since Π is irreducible, either $\Pi_i = \Pi$ or is of lower dimension. In the latter case, $\dim H + \dim \Pi_i - \dim X_i < \dim W + \dim \Pi - \dim X$, so by the induction hypothesis this term $m_i \mathrm{mdeg}_{V\times H} X_i$ does not contribute to the B-leading term of $\sum_i m_i \mathrm{mdeg}_{V\times H} X_i$.

So far we have shown that

$$\mathrm{init}_\mathrm{B}\,\mathrm{mdeg}_{V\times W} X = \mathrm{init}_\mathrm{B} \sum_i m_i\,\mathrm{mdeg}_{V\times H} X_i = \sum_i m_i\,\mathrm{init}_\mathrm{B}\,\mathrm{mdeg}_{V\times H} X_i$$

where we sum over only those X_i that project onto Π. By the induction hypothesis, $\mathrm{init}_\mathrm{B}\mathrm{mdeg}_{V\times H} X_i$ is a positive multiple of $\mathrm{B}^{\dim H + \dim \Pi - \dim X_i}\mathrm{mdeg}_V \Pi$. So every term in the

sum has B-leading term a positive multiple of $\mathrm{B}^p \operatorname{mdeg}_V \Pi$, and adding them together we get a positive multiple of $\mathrm{B}^p \operatorname{mdeg}_V \Pi$, as claimed. $\square$

In fact X was assumed irreducible only to ensure that Π is irreducible.

With more use of equivariant cohomology, one can identify the positive integer multiple as follows (a statement we neither prove nor use). Pick a general point of $\Pi \subseteq V$, and look at its preimage P in X, thought of as a subvariety of W. This preimage is automatically $\mathbb{C}^\times$-invariant, so defines a projective variety, whose degree is the desired multiple. In particular, the multiple is 1 (as it will be in our application) if and only if P is a linear subspace.

3.2. **Specializing to stabilizer subgroups.** If one defines the multidegree topologically (rather than by the inductive definition from section 1.5), the following lemma is obvious.

Lemma 2. *Let $X \subseteq V$ be a T-invariant subscheme in a T-representation V, and $\vec{v} \in V^T$ be a T-invariant vector.*

1. *The translate $X + \vec{v}$ is also T-invariant, with the same multidegree as X.*
2. *If $X \not\ni \vec{v}$, then $\operatorname{mdeg}_V X = 0$.*
3. *If X contains and is smooth at $\vec{v}$, then $\operatorname{mdeg}_V X = \prod$(the weights in $V/T_{\vec{v}}X$).*

Proof. (1) The invariance is obvious. For the equality of multidegrees, we proceed by induction on dimension. In any dimension, we can reduce to the case that X is a variety by axiom (2) of multidegrees. Hereafter we exclude the trivial case $\vec{v} = \vec{0}$ (so in particular $\dim V > 0$).

If V is one-dimensional (the base case), then it is the trivial representation, and X is either V or a finite set. Then the same is true for $X + \vec{v}$, giving the matching multidegrees 1 or 0 respectively.

If $\dim V > 1$, we can pick a T-invariant hyperplane H containing $\vec{v}$. If $H \supseteq X$, then $H \supseteq (X + \vec{v})$, and by axiom (3b) both have zero multidegree.

Otherwise $\operatorname{mdeg}_V X = \operatorname{mdeg}_H(X \cap H)$ and $\operatorname{mdeg}_V(X + \vec{v}) = \operatorname{mdeg}_H((X + \vec{v}) \cap H)$ by axiom (3a). Since $X \cap H$, $(X + \vec{v}) \cap H$ satisfy the conditions of the lemma, and are lower-dimensional schemes than X, by induction their multidegrees are the same.

(2) By part (1), we may replace X by $X - \vec{v}$, reducing to the case $\vec{v} = \vec{0}$. If $\dim V = 0$, then $X = \emptyset$, so $\operatorname{mdeg}_V X = 0$. If $\dim V > 0$, it contains a T-invariant hyperplane H. Then $X \cap H$ lives in a smaller-dimensional space but still doesn't contain $\vec{0}$, so by induction $\operatorname{mdeg}_H(X \cap H) = 0$, and $\operatorname{mdeg}_V X$ is either equal to or a multiple of that.

(3) As in part (2), we reduce to the case that $\vec{v} = \vec{0}$. By the result of part (2), we can shrink X to the union of its primary components passing through the origin. By the smoothness hypothesis, there is only one component, so X is now reduced and irreducible.

If $\dim X = 0$, then $X = \{\vec{0}\}$, and $\operatorname{mdeg}_V X$ is the product of all weights on V. If $\dim X > 0$, then we can pick a T-invariant complement to $T_{\vec{0}}X$ that is *properly* contained in V, and extend it to a T-invariant hyperplane $H \leq V$. Then $X \cap H$ is transverse at $\vec{0}$, making $X \cap H$ again smooth at the origin, so $\operatorname{mdeg}_V X = \operatorname{mdeg}_H(X \cap H) = \prod$(the weights in $H/T_{\vec{0}}(X \cap H) = \prod$(the weights in $V/T_{\vec{0}}X$), with the middle equality by induction on $\dim X$.

(The topological proofs are as follows. The family $\{X + \lambda\vec{v}\}_{\lambda\in[0,1]}$ is an equivariant homology between X and $X + \vec{v}$, so X and $X + \vec{v}$ define the same equivariant cohomology class on V. Now assuming $\vec{v} = \vec{0}$, the family $\{\lambda^{-1}X\}_{\lambda\in(0,1]}$ is an equivariant Borel-Moore homology of X to its $\lambda \to 0$ limit. If $\vec{0} \notin X$, this limit is the empty set defining the class 0. If $\vec{0} \in X$, this limit is $T_{\vec{0}}X$, whose associated class is the product indicated.) □

The following lemma (which does not involve normal cones) is stated in a somewhat roundabout way, as the proof indicates; however, this formulation is how it will actually be used in practice.

Lemma 3. *Let $X \subseteq V$ be a T-invariant subvariety of a T-representation, and $\vec{v} \in V$. Let*

$$S := \{t \in T : t\vec{v} \in \mathbb{C}\vec{v}\} \qquad \geq \qquad S_0 := \{t \in T : t\vec{v} = \vec{v}\}$$

be the projective and affine stabilizers of $\vec{v}$. Let $\mathrm{mdeg}_V^T X, \mathrm{mdeg}_V^S X, \mathrm{mdeg}_V^{S_0} X$ denote the T-, S-, and S_0-multidegrees of X.

(1) *There are natural maps $\mathrm{Sym}\, T^* \twoheadrightarrow \mathrm{Sym}\, S^* \twoheadrightarrow \mathrm{Sym}\, S_0^*$ from the specialization of variables $T^* \twoheadrightarrow S^* \twoheadrightarrow S_0^*$, and they take $\mathrm{mdeg}_V^T X \mapsto \mathrm{mdeg}_V^S X \mapsto \mathrm{mdeg}_V^{S_0} X$.*
(2) *If $\vec{v} \notin X$, then $\mathrm{mdeg}_V^S X$ is a multiple of the S-weight on $\vec{v}$, and $\mathrm{mdeg}_V^{S_0} X = 0$.*
(3) *If $\vec{v}$ is a smooth point in X, then $\mathrm{mdeg}_V^{S_0} X = \prod$ (the S_0-weights in $V/T_{\vec{v}}X$).*

Proof. The image of $\mathrm{mdeg}_V X$ in $\mathrm{Sym}\, S^*$ is just the S-multidegree. So we may as well restrict to the S-action from the beginning. By axiom (2) of multidegrees, we may assume X is reduced and irreducible.

The first statement follows trivially from the axioms inductively defining multidegrees.

Let L be a one-dimensional S-invariant subspace containing $\vec{v}$. (If $\vec{v} \neq \vec{0}$, then L is of course unique.) Let H be an S-invariant complementary hyperplane. Let $\Pi \subseteq H$ be the closure of the image of the projection of X, and d the degree of the projection map $X \to \Pi$.

Now apply [KMY, Theorem 2.5], which says that $\mathrm{mdeg}_V X = d\ \mathrm{mdeg}_V \Pi$, plus a term that vanishes under the assumption $\mathbb{C}\vec{v} \not\subset X$. Then by property (3a) of multidegrees, $\mathrm{mdeg}_V \Pi$ is a multiple of the S-weight on L, and its S_0-multidegree is a multiple of the S_0-weight on L, which is zero.

The third statement is just an application of Lemma 2. □

This result gives the most information when S is as large as possible, which means $\vec{v}$ is very special. Unfortunately it seems that it is then very likely to be in X, and not be a smooth point. So we give a criterion for this smoothness:

Lemma 4. *Let V be a scheme carrying an action of a group G. (V need not be a vector space.) Let $Y \subseteq V$ be an irreducible subvariety, and let $X = \overline{G \cdot Y}$ (with the reduced scheme structure).*

Then a generic point $y \in Y$ is a smooth point of X, and its tangent space T_yX is $\mathfrak{g} \cdot T_yY$.

Proof. Assume the contrary: then every point in Y is a singular point of X. Since G acts by automorphisms, all of $G \cdot Y$ consists of singular points in X, and by semicontinuity X consists only of singular points. But since X is reduced, its smooth locus is open dense, a contradiction.

Let $Y^\circ \subseteq Y$ be the open subset of $y \in Y$ with the minimum dimensional G-stabilizer (among points in Y). Hence y is not in the closure of any G-orbit on Y. Our genericity

condition on y will be that y is a smooth point of Y°. So $T_y X = T_y(G \cdot Y^\circ)$. By the definition of Y°, the map $G \times Y^\circ \twoheadrightarrow G \cdot Y^\circ$ is a submersion, enabling us to compute the tangent spaces of the target. $\square$

Proposition 4. *Let V be a representation of a group G, with Y a subspace fixed pointwise by a torus S_0, i.e. $Y \leq V^{S_0}$. Then*

$$\mathrm{mdeg}_V^{S_0} \overline{G \cdot Y} \neq 0 \qquad \iff \qquad \overline{G \cdot Y} = \overline{G \cdot V^{S_0}} \qquad \iff \qquad \overline{G \cdot Y} \supseteq V^{S_0} \quad \textit{(e.g. if } Y = V^{S_0}\textit{).}$$

More specifically, $\mathrm{mdeg}_V^{S_0} \overline{G \cdot Y} = \prod$ *(the S_0-weights in $V/(\mathfrak{g} \cdot Y)$).*

Proof. Since $Y \leq V^{S_0}$, we know $\overline{G \cdot Y} \subseteq \overline{G \cdot V^{S_0}}$. In general, $\overline{G \cdot A} \supseteq B$ iff $\overline{G \cdot A} \supseteq \overline{G \cdot B}$. Together these establish the equivalence of the latter two conditions.

The first condition implies the third by Lemma 3 part 2. All that remains for the equivalences is to show the second implies the first.

Before doing that, we compute $\mathrm{mdeg}_V^{S_0} \overline{G \cdot Y}$ exactly, using Lemma 3, part 3, with $\vec{v} = y$; this gives the claimed formula.

The S_0-weights in $V/(\mathfrak{g} \cdot Y)$ form a subset of the S_0-weights in V/Y, and the condition $Y = V^{S_0}$ says that none of these are zero, so the product is nonzero. $\square$

We will actually use very little of Proposition 4; we felt it was worth including because many of the varieties for which multidegrees have been computed are of this form $\overline{G \cdot V^{S_0}}$ (e.g. [KZJ07, KM05]).

3.3. **Application to the Brauer loop scheme.** For simplicity, we assume N to be even, $N = 2n$.

Recall from the introduction that $R_{\mathbb{Z} \bmod N}$ denotes the algebra of infinite periodic upper triangular matrices, and contains R_N as a subalgebra in a natural way. Let $R^\times_{\mathbb{Z} \bmod N} \geq R^\times_N = B$ denote their multiplicative groups, which consist of those matrices with nonzero diagonal entries. Then the conjugation action of $R^\times_{\mathbb{Z} \bmod N}$ on $R_{\mathbb{Z} \bmod N}$ descends to an action on the Brauer loop scheme.

Consequently the projection map in section 1.2, from the Brauer loop scheme to the orbital scheme, is B-equivariant. This fact was used in [KZJ07] to determine the top-dimensional components of the Brauer loop scheme.

Theorem 7. *Let $\pi \in S_N$ be an involution with no fixed points, and E_π the corresponding Brauer loop variety. Then the dimension of the projection of E_π to $\overline{B \cdot \pi_<}$ is $n^2 - c$, where c is the number of crossings in π's chord diagram.*

The B-leading form of the Brauer loop polynomial Ψ_π is $B^{n^2-n-c} J_\pi$, where J_π is the corresponding Joseph–Melnikov polynomial.

Proof. Let Π denote the image of the projection. By the B-equivariance, $B \cdot \pi_< \subseteq \Pi$, and since Π is closed, $\overline{B \cdot \pi_<} \subseteq \Pi$. Then we apply Theorem 4.

For the latter statement, we use the decomposition of $\mathcal{M}_N^{\Delta=0}$ into R_N plus its unique T-invariant complement; these will be the V and W of Proposition 3. The predicted exponent $\dim W + \dim \Pi - \dim X$ is then $\binom{2n}{2} + (n^2 - c) - 2n^2 = n^2 - n - c$. $\square$

Example. We list the Brauer loop polynomials in size $N = 2n = 4$:

$$
\begin{aligned}
\Psi_{(12)(34)} =&(A + z_2 - z_3)(B + z_4 - z_1)\\
&(A^2 + 2AB + Bz_1 - Az_2 - z_1z_2 + Az_3 + z_2z_3 - Bz_4 + z_1z_4 - z_3z_4)\\
\Psi_{(14)(23)} =&(A + z_1 - z_2)(A + z_3 - z_4)\\
&(A^2 + AB + B^2 - Bz_1 + Az_2 + z_1z_2 - Az_3 - z_2z_3 + Bz_4 - z_1z_4 + z_3z_4)\\
\Psi_{(13)(24)} =&(A + z_1 - z_2)(A + z_2 - z_3)(A + z_3 - z_4)(B + z_4 - z_1)
\end{aligned}
$$

(As the factorizations suggest, only the last one corresponds to a complete intersection, with the obvious linear equations.) We leave to the reader to check that the B-leading forms are the Joseph–Melnikov polynomials of the example of the previous section. Note that the cyclic shift of indices $z_i \mapsto z_{i+1}$ (recall that $z_{i+N} = z_i + A - B$) exchanges the first two polynomials and leaves the third invariant.

Proposition 5. *(see* [DFZJ06, Lemma 2]*) Let* π, ρ *be link patterns. Then the specialization of* Ψ_π *under the identifications* $\{A = B = 0, y_i = y_{\rho(i)}\}$ *is zero unless* $\pi = \rho$*, in which case it is nonzero.*

Consequently, the polynomials $\{\Psi_\pi\}$ *are linearly independent over* $\mathbb{Z}$*.*

Proof. Let $Y_\pi = \overline{\pi T}$, the vector space of matrices with entries only in the same positions as the permutation matrix π. This Y_π is T-stable, and fixed pointwise by $S_0^\pi := \{\mathrm{diag}(\dots, t_i, \dots) : t_i = t_{\pi(i)}\}$. By [KZJ07, Theorem 3], $E_\pi = \overline{R^\times_{\mathbb{Z} \bmod N} \cdot Y}$.

Similarly define Y_ρ and S_0^ρ, and let M be a generic element of Y_ρ. First we prove the vanishing, where $\rho \neq \pi$. By [KZJ07, Theorem 5 part 2], $M \notin E_\pi$. Then apply Lemma 3 part 2 to see that Ψ_π vanishes under the joint specialization $\{A = 0, y_i = y_{\rho(i)}\}$.

For the nonvanishing when $\pi = \rho$, we apply Proposition 4.

Linear independence over $\mathbb{Z}$ is then proved by the standard argument: take a purported linear relation and specialize at $\{A = B = 0, y_i = y_{\rho(i)}\}$ to isolate the coefficient of Ψ_π. Repeat for each π. ☐

Though we won't need it, it is not hard to actually compute the specialization of Ψ_π at $\{A = B = 0, y_i = y_{\rho(i)}\}$ (and is essentially done in [DFZJ06, Lemma 2]); it is the product $\prod(y_i - y_j)$ over entries (i, j) in the "diagram" of the affine permutation π, as defined at the end of section 3 of [KZJ07]. (It should have been called the "Rothe diagram" there, to distinguish it from the chord diagram.)

4. Interlude: Polynomial solution of the Brauer qKZ equation

This section is independent from sections 2, 3. It might thus seem that we use the same notation for a priori unrelated quantities; but this is made in order to facilitate the identification that will be made in section 5.

4.1. **The Brauer algebra.** As before, let N be an even positive integer, $N = 2n$. The **Brauer algebra** is a $\mathbb{C}[\beta]$-algebra defined by generators f_i and e_i, $i = 1, \dots, N-1$, and relations

$$
\begin{aligned}
&e_i^2 = \beta e_i \qquad e_i e_{i\pm1} e_i = e_i \\
&f_i^2 = 1 \qquad (f_i f_{i+1})^3 = 1 \\
&f_i e_i = e_i f_i = e_i \quad e_i f_i f_{i+1} = e_i e_{i+1} = f_{i+1} f_i e_{i+1} \quad e_{i+1} f_i f_{i+1} = e_{i+1} e_i = f_i f_{i+1} e_i \\
&e_i e_j = e_j e_i \qquad f_i f_j = f_j f_i \qquad e_i f_j = f_j e_i \qquad |i-j| > 1
\end{aligned}
\tag{4.1}
$$

where indices take all values for which the identities make sense. Note that the f_i are generators of the symmetric group $\mathcal{S}_N$.

For each $i = 1, \dots, N-1$ define the R-matrix

$$
\check{R}_i(u) = \frac{A(A-u) + A\,u\,e_i + (1-\beta/2)u(A-u)f_i}{(A+u)(A-(1-\beta/2)u)}
\tag{4.2}
$$

Here u and A are formal parameters.

Using the defining relations of the Brauer algebra, one can show that the $\check{R}_i$ satisfy the *Yang–Baxter equation*

$$
\check{R}_i(u)\check{R}_{i+1}(u+v)\check{R}_i(v) = \check{R}_{i+1}(v)\check{R}_i(u+v)\check{R}_{i+1}(u) \qquad i = 1, \dots, N-2
\tag{4.3}
$$

and the unitarity equation

$$
\check{R}_i(u)\check{R}_i(-u) = 1 \qquad i = 1, \dots, N-1.
\tag{4.4}
$$

Finally we introduce a representation space V for the Brauer algebra. V has as a canonical basis the involutions of $\{1, \dots, N\}$ without fixed points; we will refer to them as **link patterns**, and draw them as pairings of points in the upper-half plane, in such a way that they cross transversely and at most once; such "crossings" in a link pattern only appear between the pairings $(i, \pi(i))$ and $(j, \pi(j))$ if $i < j < \pi(i) < \pi(j)$ up to switching $i \leftrightarrow \pi(i)$, $j \leftrightarrow \pi(j)$, $(i, \pi(i)) \leftrightarrow (j, \pi(j))$. Sometimes it will be convenient to identify $\{1, \dots, N\}$ with $\mathbb{Z}/N\mathbb{Z}$, in which case one can alternatively draws link patterns as pairings of points in a disc, with the N points placed in the counterclockwise order on the boundary circle (as was done in [KZJ07]).

The representation of the Brauer algebra on $V[\beta]$ can be described by the action of its generators on the canonical basis. The generator f_i acts on a link pattern π as the transposition $(i, i+1)$ acting by conjugation, whereas e_i acts as

$$
e_i \cdot \pi = \begin{cases} \beta\,\pi & \text{if } \pi(i) = i+1 \\ \pi' & \text{otherwise, where } \pi' \text{ has cycles } (i, i+1),\ (\pi(i), \pi(i+1)) \text{ and the rest like } \pi \end{cases}
\tag{4.5}
$$

This action is best understood graphically, see Fig. 3.

We need one more operator r acting on V: it conjugates by the cycle $(12\cdots N)$. To avoid confusion we *do not* identify r with the corresponding abstract element of the symmetric group (see the remark at the end of appendix A). If link patterns are drawn on a circle, then r is the operator that rotates them one step counterclockwise.

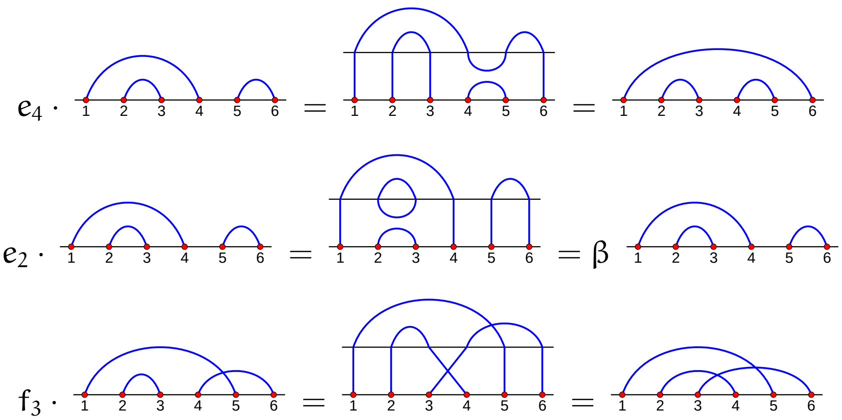


FIGURE 3. Link patterns and Brauer algebra generators acting on them.

4.2. **Brauer qKZ equation.** Let us now introduce one more parameter ϵ and consider the following set of equations:

$$\check{R}_i(z_i - z_{i+1})\Psi_N(z_1, \dots, z_N) = \Psi_N(z_1, \dots, z_{i+1}, z_i, \dots, z_N) \qquad i = 1, \dots, N-1 \tag{4.6}$$

$$r^{-1}\Psi_N(z_1, \dots, z_N) = \Psi_N(z_2, \dots, z_N, z_1 + \epsilon) \tag{4.7}$$

where Ψ_N belongs to $V[A, \epsilon, z_1, \dots, z_N, \beta]$, $\Psi_N = \sum_\pi \Psi_\pi \pi$.

On general grounds, one does not expect that there exist polynomial solutions of the system of equations (4.6–4.7) – that is, for generic values of the parameters β, A, ϵ. We shall see below that such a solution exists if $\beta = \frac{2(A-\epsilon)}{2A-\epsilon}$, i.e. if one considers the equation in the quotient $V[A, \epsilon, z_1, \dots, z_N, \beta]/\langle(2A - \epsilon)\beta - 2(A - \epsilon)\rangle$. But first we need to introduce some convenient notation and reformulations of the qKZ equation.

We extend the variables $z_1, \dots, z_N$ to an infinite set $\{z_i, i \in \mathbb{Z}\}$ by imposing the pseudo-cyclicity condition $z_{i+N} = z_i + \epsilon$. Let us call $W = \mathbb{C}[A, \epsilon, z_i, i \in \mathbb{Z}, \beta]/\langle z_{i+N} - z_i - \epsilon, i \in \mathbb{Z}\rangle$. Let τ_i be the operator appearing in the r.h.s. of Eq. (4.6), that is in our periodic notations the automorphism of W that exchanges variables z_{i+kN} and z_{i+1+kN} for all $k \in \mathbb{Z}$. Similarly, define for future use σ to be the operator appearing in the r.h.s. of Eq. (4.7), that is the automorphism of W that shifts $z_i \mapsto z_{i+1}$ for all $i \in \mathbb{Z}$. Finally, introduce generators of the *affine* Brauer algebra e_i, f_i for all $i \in \mathbb{Z}/N\mathbb{Z}$ that satisfy the same relations (4.1) as the Brauer algebra, but with indices modulo N. In particular the f_i are generators of the affine symmetric group $\hat{\mathcal{S}}_N$. The representation on V of the Brauer algebra can be extended to the affine Brauer algebra in a natural way (f_N being conjugation by the transposition $(1, N)$ and e_N creating cycles $(1, N)$, $(\pi(1), \pi(N))$), in such a way that (as operators on V) they satisfy $e_{i+1} = r e_i r^{-1}$ and $f_{i+1} = r f_i r^{-1}$ for all $i \in \mathbb{Z}/N\mathbb{Z}$. Then we have the

Proposition 6. *If Ψ_N is solution of Eqs. (4.6–4.7), then*

$$\check{R}_i(z_i - z_{i+1})\Psi_N = \tau_i \Psi_N \qquad i \in \mathbb{Z} \tag{4.8}$$

Proof. If $i \neq 0 \mod N$, then Eq. (4.8) is nothing but Eq. (4.6). Furthermore, our new notations make the following formulae valid: $\check{R}_{i+1}(z_{i+1} - z_{i+2}) = \sigma r \check{R}_i(z_{i+1} - z_i) r^{-1} \sigma^{-1}$ and $\tau_{i+1} = \sigma \tau_i \sigma^{-1}$. Eq. (4.7), namely $r\sigma\Psi_N = \Psi_N$, then allows us to shift the index i of Eq. (4.6) to arbitrary values. $\square$

We need to rewrite more explicitly Eqs. (4.8). Expanding in the canonical basis, we find that for a given link pattern π, the equation involves the preimage of π under e_i and f_i. We are led to the following dichotomy:

- either $\pi(i) \neq i+1$, in which case the preimage of π under e_i is empty, and a small calculation results in

(4.9)
$$\Psi_{f_i \cdot \pi} = \Theta_i \Psi_\pi \qquad \Theta_i := \frac{(A + z_i - z_{i+1})(A + (1-\beta/2)(z_{i+1} - z_i))\tau_i - A(A - z_i + z_{i+1})}{(1-\beta/2)(z_i - z_{i+1})(A - z_i + z_{i+1})}$$

- or $\pi(i) = i+1$, in which case $f_i \cdot \pi = \pi$ and we find this time

$$\sum_{\pi' \neq \pi, e_i \cdot \pi' = \pi} \Psi_{\pi'} = \Delta_i \Psi_\pi \qquad \Delta_i := \frac{(A + z_i - z_{i+1})(A + (1-\beta/2)(z_{i+1} - z_i))(\tau_i - 1)}{A(z_i - z_{i+1})} \tag{4.10}$$

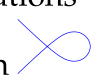
The Θ_i and Δ_i are operators on $\widetilde{W} := W[(1-\beta/2)^{-1}, A^{-1}, (A - z_i + z_{i+1})^{-1}, (z_i - z_{i+1})^{-1}, i \in \mathbb{Z}]$ with the following properties:

Lemma 5. *The Θ_i, $i \in \mathbb{Z}$, satisfy the affine symmetric group relations*

$$\Theta_i^2 = 1 \qquad (\Theta_i \Theta_{i+1})^3 = 1 \qquad \Theta_i \Theta_j = \Theta_j . \Theta_i \quad |i-j| > 1 \tag{4.11}$$

In what follows we shall be particularly interested in the equations (4.9) involving f_i and Θ_i. In view of Lemma 5, one can define a group morphism $\Theta : s \mapsto \Theta_s$ from $\hat{\mathcal{S}}_N$ to the invertible operators on $\widetilde{W}$ such that $f_i \mapsto \Theta_i$. Note however that the action of Θ_i on Ψ_π only makes sense when $\pi(i) \neq i+1$. We are therefore led to a more relevant groupoid structure. Define

$$\hat{\mathcal{S}}_{\pi,\pi'} := \{ s \in \hat{\mathcal{S}}_N, s \cdot \pi = \pi' : \exists (i_1, \ldots, i_k) \text{ such that } \\ s = f_{i_k} \cdots f_{i_1} \text{ and } \forall \ell = 1, \ldots, k \ (f_{i_{\ell-1}} \cdots f_{i_1} \cdot \pi)(i_\ell) \neq i_\ell + 1 \} \tag{4.12}$$

Graphically, $\hat{S}_{\pi,\pi'}$ is the set of "affine permutations" that map π to π' without creating "tadpoles", i.e. lines that cross themselves as in .

Proposition 7. *The $(\hat{\mathcal{S}}_{\pi,\pi'})$ form a groupoid, and $\hat{\mathcal{S}}_{\pi,\pi'} \neq \emptyset \ \forall \pi, \pi'$; Θ is a groupoid morphism; and if Ψ_N satisfies Eqs. (4.9),*

$$\Psi_{\pi'} = \Theta_s \Psi_\pi \qquad \forall \pi, \pi' \textit{ and } \forall s \in \hat{\mathcal{S}}_{\pi,\pi'} \tag{4.13}$$

Proof. The only non-trivial statement is that $\hat{\mathcal{S}}_{\pi,\pi'} \neq \emptyset \ \forall \pi, \pi'$. Indeed we shall show that $\mathcal{S}_{\pi,\pi'} \neq \emptyset$, where $\mathcal{S}_{\pi,\pi'} = \hat{\mathcal{S}}_{\pi,\pi'} \cap \mathcal{S}_N$. As $\mathcal{S}_N$ acts transitively by conjugation on involutions without fixed points, we may pick $s_0 \in \mathcal{S}_N$ such that $s_0 \cdot \pi = \pi'$, and any decomposition of it in terms of generators: $s_0 = f_{i_k} \cdots f_{i_1}$. Consider the new word obtained from $f_{i_k} \cdots f_{i_1}$ by removing each f_{i_ℓ} such that $(f_{i_{\ell-1}} \cdots f_{i_1} \cdot \pi)(i_\ell) = i_\ell + 1$. Since in this case $f_{i_\ell} f_{i_{\ell-1}} \cdots f_{i_1} \cdot \pi = f_{i_{\ell-1}} \cdots f_{i_1} \cdot \pi$, the new word defines a permutation s such that $s \cdot \pi = \pi'$ but now also satisfies the defining property of $\hat{\mathcal{S}}_{\pi,\pi'}$; which is therefore non-empty. ☐

Proposition 8. *Let Ψ be a solution of Eqs. (4.8) and π a link pattern. If $i, j \in \mathbb{Z}$, $i < j$, are such that $\pi(\{i, i+1, \ldots, j\}) \cap \{i, i+1, \ldots, j\} = \emptyset$ (mod N is implied), then $A + z_i - z_j$ divides Ψ_π.*

Proof. Induction on $j-i$.

Rewrite Eq. (4.8) as $\Psi_N = \check{R}_i(z_{i+1}-z_i)\tau_i\Psi_N$, and look at its component π, noting that $\check{R}_i(z_{i+1}-z_i) = (A+z_i-z_{i+1})(F_1+F_2 f_i)+F_3 e_i$ where F_1, F_2, F_3 are some rational fractions of z_i, z_{i+1}, A without pole at $z_{i+1} = A+z_i$. The hypothesis implies $\pi(i) \neq i+1$, i.e $\pi \notin \operatorname{Im} e_i$, so that we can ignore the third term.

If $j-i=1$, we conclude directly that $A+z_i-z_{i+1} \,|\, \Psi_\pi$.

If $j-i>1$, we note that both π and $f_i\cdot\pi$ satisfies the hypotheses for the interval $\{i+1,\dots,j\}$ so that we can use the induction to conclude that $A+z_{i+1}-z_j$ divides both Ψ_π and $\Psi_{f_i\cdot\pi}$, or equivalently that $A+z_i-z_j$ divides both $\tau_i\Psi_\pi$ and $\tau_i\Psi_{f_i\cdot\pi}$, hence also Ψ_π. ☐

4.3. **Solution of qKZ equation.** We can now state the main theorem of this section:

Theorem 8. *If $\beta = \frac{2(A-\epsilon)}{2A-\epsilon}$, there is a solution of Eqs. (4.6–4.7) such that its components Ψ_π are homogeneous polynomials in $\mathbb{Z}[A,\epsilon,z_1,\dots,z_N]$, of degree $2n(n-1)$. This solution is unique up to numerical normalization.*

Proof. The proof will be similar to the statement of [DFZJ06] that it generalizes. First, introduce the "base" link pattern π_0: $\pi_0(i) = i+n, \quad i=1,\dots,n$. Applying Prop. 8, we find a product of $2n(n-1)$ factors in Ψ_{π_0}. To ensure the correct degree, we must impose (up to numerical normalization):

$$\Psi_{\pi_0} = \prod_{i=1}^{N}\prod_{j=i+1}^{i+n-1}(A+z_i-z_j) \tag{4.14}$$

or in terms of the original variables, $\Psi_{\pi_0} = \prod_{\substack{1\le i<j\le 2n\\ j-i<n}}(A+z_i-z_j)\prod_{\substack{1\le i<j\le 2n\\ j-i>n}}(B+z_j-z_i)$, with the convenient notation $B := A-\epsilon$.

Next we have the following elementary lemma:

Lemma 6. Ψ_N *is determined uniquely by Eq. (4.14) and Eqs. (4.9).*

Proof. If Eqs. (4.9) are satisfied, we can apply Prop. 7 and state that

$$\Psi_\pi = \Theta_s\Psi_{\pi_0} \qquad \forall\pi \text{ and } \forall s\in\hat{\mathcal{S}}_{\pi_0,\pi} \tag{4.15}$$

with $\hat{\mathcal{S}}_{\pi_0,\pi}\neq\emptyset$. Thus, if Ψ_{π_0} is fixed by Eq. (4.14), all Ψ_π are fixed. ☐

Now that we have proved uniqueness, we want to show existence of a solution. We thus consider Eqs. (4.14–4.15) as *defining* the entry Ψ_π, and all we need to prove is that this definition is independent of the choice of s. Equivalently we need to show that

$$\Psi_{\pi_0} = \Theta_s\Psi_{\pi_0} \qquad \forall s\in\hat{\mathcal{S}}_{\pi_0,\pi_0}. \tag{4.16}$$

We need the additional

Lemma 7. $\hat{\mathcal{S}}_{\pi_0,\pi_0}$ *is the subgroup of $\hat{\mathcal{S}}_N$ generated by the $f_i f_{i+n}$, $i\in\mathbb{Z}/N\mathbb{Z}$.*

The proof is in Appendix A.

We thus need to show that $\Theta_i\Psi_{\pi_0} = \Theta_{i+n}\Psi_{\pi_0}$, which we do using the following alternative formula for Θ_i:

$$\Theta_i = -1-\left(\frac{A}{1-\beta/2}+z_{i+1}-z_i\right)(A+z_i-z_{i+1})\partial_i\frac{1}{A+z_i-z_{i+1}} \qquad \partial_i := \frac{1}{z_i-z_{i+1}}(1-\tau_i) \tag{4.17}$$

This ∂_i is a **divided difference operator**; it has the important property that $\partial_i(fg) = f\,\partial_i g$ if f is symmetric in z_i, z_{i+1} (equivalently, if $\partial_i f = 0$). From Eq. (4.14) we deduce that

$$\Psi_{\pi_0} = (A + z_i - z_{i+1})(A + z_{i+1} - z_{i+n})(A + z_{i+1+n} - \epsilon - z_i)(A + z_{i+n} - z_{i+n+1}) \times S$$

where S is symmetric in both $z_i \leftrightarrow z_{i+1}$ and $z_{i+n} \leftrightarrow z_{i+n+1}$. Performing the computation we find

$$\Theta_i\Psi_{\pi_0} - \Theta_{i+n}\Psi_{\pi_0} = \frac{-A + A\beta + \epsilon - \beta\epsilon/2}{1-\beta/2}(A+z_i-z_{i+1})(A+z_{i+n}-z_{i+n+1})(z_i-z_{i+1}-z_{i+n}+z_{i+n+1})S$$

which is zero if the condition $\beta = \frac{2(A-\epsilon)}{2A-\epsilon}$ is satisfied.

Thus, Ψ_N is well-defined. Furthermore, one notes that when one computes Ψ_π using Eq. (4.15), i.e. by acting with successive Θ_{i_ℓ}, due to the definition of $\hat{\mathcal{S}}_{\pi_0,\pi}$, each Θ_{i_ℓ} acts on a $\Psi_{\pi'}$ (with $\pi' = f_{i_{\ell-1}} \cdots f_{i_\ell} \cdot \pi_0$) such that $\pi'(i_\ell) \neq i_\ell + 1$. Therefore we can apply Prop. 8, which says that $A + z_{i_\ell} - z_{i_\ell+1} \mid \Psi_{\pi'}$, and conclude using the alternative form (4.17) for Θ_i (with $A/(1-\beta/2) = A + B$) that the polynomial character of $\Psi_\pi(z_1, \ldots, z_N, A, B)$ and integrality of its coefficients are preserved by the successive actions of the Θ_i. Since the operators Θ_i are also degree-preserving, all Ψ_π are of degree $2n(n-1)$.

We now want to show that Ψ is a solution of Eqs. (4.6–4.7).

Eq. (4.7) written in components is $\Psi_{r\cdot\pi} = \sigma\Psi_\pi$.

Using $\Theta_{i+1} = \sigma\Theta_i\sigma^{-1}$, we find that if $\Psi_\pi = \Theta_s\Psi_{\pi_0}$, $\Psi_{r\cdot\pi} = \sigma\Theta_s\sigma^{-1}\Psi_{r\cdot\pi_0}$. All that needs to be checked is that $r \cdot \pi_0 = \pi_0$, which is obvious from its definition, and that $\Psi_{\pi_0} = \sigma\Psi_{\pi_0}$, which follows from the explicitly rotation-invariant expression (4.14).

To check Eq. (4.6), we consider separately the two cases of Eqs. (4.9) and (4.10). Eq. (4.9), and more generally Eq. (4.13), are tautologies with our definition (4.15) of Ψ_π: they express the fact that Θ is a groupoid morphism.

Eq. (4.10) requires a calculation. By the usual rotational invariance argument ($re_ir^{-1} = e_{i+1}$ and $\sigma\Delta_i\sigma^{-1} = \Delta_{i+1}$), one can assume that $i = 1$. The f_j, $j = 3, \ldots, N-1$ generate a subgroup $\mathcal{S}_{N-2}$ which acts transitively by conjugation on the link patterns such that $\pi(1) = 2$ (i.e. involutions without fixed points of $N-2$ elements). By the same argument as in the proof of Prop. 7, for any pair of such involutions $\hat{\mathcal{S}}_{\pi,\pi'} \cap \mathcal{S}_{N-2} \neq \emptyset$, and we thus have $s \in \mathcal{S}_{N-2}$ such that $\Psi_{\pi'} = \Theta_s\Psi_\pi$. Act on both sides of Eq. (4.10) with Θ_s. Using $\Delta_1\Theta_j = \Theta_j\Delta_1$ and $e_1^{-1}(f_j(\pi)) = f_j(e_1^{-1}(\pi))$ for $j \neq 1, 2, N$, we deduce that equations (4.10) for any π and π' are equivalent. One can then check it for a special case, the simplest being the following: choose π with cycles $(1,2)$ and $(j, j+n-1)$, $j = 3, \ldots, n+1$. Noting that link patterns in $e_1^{-1}(\pi)$ come in pairs π_j, $f_i \cdot \pi_j$, $j = 3, \ldots, n+1$, where $\pi_j(1) = j$ and $\pi_j(2) = j + n - 1$, we conclude that $\sum_{\pi' \neq \pi, e_1\cdot\pi' = \pi} \Psi_{\pi'} = (1 + \Theta_1)\sum_{j=3}^{n+1} \Psi_{\pi_j}$. Using the expression (4.17) for Θ_1, we find that Eq. (4.10) is equivalent to the fact that $\Phi := \Psi_\pi - \frac{2A-\epsilon}{A+z_1-z_2}\sum_{j=3}^{n+1}\Psi_{\pi_j}$ is symmetric in z_1, z_2. It is left to the reader to check that the following expression holds:
(4.18)

$$\Phi = \prod_{\substack{3\le i<j\le 2n \\ j-i<n-1}} (A+z_i-z_j) \prod_{\substack{3\le i<j\le 2n \\ j-i>n-1}} (B+z_j-z_i) \prod_{j=3}^{n+1}(A+z_1-z_j)(A+z_2-z_j) \prod_{i=n+2}^{2n} (B+z_i-z_1)(B+z_i-z_2)$$

(we recall that $B = A - \epsilon$) so that in particular it is symmetric in z_1, z_2. □

For later reference, we restate equations (4.9,4.10) with the specialization $\beta = \frac{2(A-\epsilon)}{2A-\epsilon}$ and $B = A - \epsilon$, making $1 - \beta/2 = A/(A+B)$. We will also clear denominators, some of which can be subsumed into the **divided difference operator** ∂_i, defined by $\partial_i p := (p - \tau_i \cdot p)/(z_i - z_{i+1})$. In the same dichotomy,

- if $\pi(i) \neq i+1$:

$$\Psi_{f_i\cdot\pi} = \Theta_i \Psi_\pi \qquad \Theta_i := \frac{(A + z_i - z_{i+1})(A + (1-\beta/2)(z_{i+1} - z_i))\tau_i - A(A - z_i + z_{i+1})}{(1-\beta/2)(z_i - z_{i+1})(A - z_i + z_{i+1})}$$

  which one can rewrite equivalently:

$$(A + B + z_{i+1} - z_i)(A + z_i - z_{i+1})(-\partial_i)\frac{\Psi_\pi}{A + z_i - z_{i+1}} = \Psi_\pi + \Psi_{f_i\cdot\pi}. \tag{4.19}$$

- if $\pi(i) = i+1$:

$$(A + B + z_{i+1} - z_i)(A + z_i - z_{i+1})(-\partial_i)\Psi_\pi = (A+B) \sum_{\pi' \neq \pi, e_i\cdot\pi' = \pi} \Psi_{\pi'} \tag{4.20}$$

5. Geometric interpretation of the Brauer action

In [KZJ07] we gave a geometric derivation of the action of the f_i generators of the Brauer algebra on the vector space spanned by the Brauer loop polynomials $\{\Psi_\pi\}$ (at $A = B$). Using algebraic results of [DFZJ06], we showed that this implied an action of the e_i generators, but did not give a geometric interpretation.

In this section we review first Hotta's construction of the action of the e_i generators on the vector space spanned by Joseph polynomials. We also discuss the analogue of the f_i action on Joseph–Melnikov polynomials.

We then move beyond the orbital scheme to the Brauer loop scheme. We review the action of the f_i generators from [KZJ07]. This leads us to our main result, identifying the solution of qKZ equation discussed in the previous section with the multidegrees of the irreducible components of the Brauer loop scheme. We then give a direct geometric interpretation of the action of the e_i generators on the vector space spanned by Brauer loop polynomials.

5.1. Hotta's construction of Springer representations.

Let D be the closure of a nilpotent orbit of $GL_N(\mathbb{C})$ (though Hotta's construction works for other groups as well), and $D \cap R_N$ the corresponding orbital scheme, with components $\{D_\tau\}$. Note that each $D_\sigma \subseteq R_N^{\Delta=0}$, i.e. the diagonal of a nilpotent upper triangular matrix vanishes. Fix a particular orbital variety, D_σ, which we recall to be automatically B-invariant.

Fix $i \in \{1, 2, \ldots, N-1\}$. There is a corresponding GL_2 subgroup of $GL_N(\mathbb{C})$, consisting of matrices M that look like the identity matrix except in entries M_{ab} with $a, b \in \{i, i+1\}$. Call this subgroup $(GL_2)_i$, and let $B_i := (GL_2)_i \cap B$.

5.1.1. *Cutting then sweeping.*

There are two cases. If every $M \in D_\sigma$ has $M_{i,i+1} = 0$, then D_σ is $(GL_2)_i$-invariant. This is the boring case.

Otherwise $D_\sigma \cap \{M_{i,i+1} = 0\}$ is codimension 1 in D_σ, and is itself B-invariant. Let its geometric components be $\{D'_\tau\}$ (the precise indexation being unspecified yet) appearing

with multiplicities $\{m_{\sigma\tau} \in \mathbb{N}\}$. Each such D'_τ is B-invariant hence B_i-invariant, so

$$\dim((GL_2)_i \cdot D'_\tau) \le \dim D'_\tau + \dim((GL_2)_i/B_i) = (\dim(D \cap R_N) - 1) + 1 = \dim(D \cap R_N).$$

Plainly $(GL_2)_i \cdot D'_\tau \subseteq D$, and it is also easy to see that $(GL_2)_i \cdot D'_\tau \subseteq R_N$. So if the above dimension inequality is tight, $(GL_2)_i \cdot D'_\tau$ must again be an orbital variety, say D_τ (this fixes the indexation of the D'_τ).

Having pursued the geometry, we now give the corresponding multidegree calculation. We introduce to that effect to the $(GL_2)_i$-invariant hyperplane

$$R_N^- := R_N^{\Delta=0} \cap \{M_{i,i+1} = 0\}$$

and use the equality (property 3(a) from §1.5) $\mathrm{mdeg}_{R_N^{\Delta=0}} D_\sigma = \mathrm{mdeg}_{R_N^-}(D_\sigma \cap \{M_{i,i+1} = 0\}$. Cutting with $\{M_{i,i+1} = 0\}$ results in the decomposition

$$\mathrm{mdeg}_{R_N^{\Delta=0}} D_\sigma = \mathrm{mdeg}_{R_N^-}(D_\sigma \cap \{M_{i,i+1} = 0\}) = \sum_\tau m_{\sigma\tau}\, \mathrm{mdeg}_{R_N^-} D'_\tau. \tag{5.1}$$

To understand the effect of sweeping out a B_i-invariant variety using $(GL_2)_i$, we need the following special case of a result from [Jo84, BBM89], spelled out in the present language in [KZJ07, Lemma 1].

Lemma 8. *Let* $V \le M_N(\mathbb{C})$ *be a subspace invariant under* B *and* $(GL_2)_i$. *Let* X *be a variety in* V *invariant under* B *and rescaling, with multidegree* $\mathrm{mdeg}_V X$. *If the generic fiber of the map*

$$\mu : ((GL_2)_i \times X)/B_i \to V, \qquad [g, x] \mapsto g \cdot x$$

is finite over Image μ, *call its cardinality* k; *otherwise let* $k = 0$. *(The latter occurs iff* X *is* $(GL_2)_i$*-invariant.) Then*

$$k\, \mathrm{mdeg}_V(\mathrm{Image}\ \mu) = -\partial_i\, \mathrm{mdeg}_V X$$

where ∂_i *is the divided difference operator,* $\partial_i p = (p - r_i \cdot p)/(z_i - z_{i+1})$, *defined using the reflection* r_i *that switches* $z_i \leftrightarrow z_{i+1}$.

Applying the lemma amount to applying $-\partial_i$ to Eq. (5.1). Noting that the right hand side $\sum_\tau m_{\sigma\tau}\, \mathrm{mdeg}_{R_N^-} D'_\tau$ becomes

$$\sum_\tau m_{\sigma\tau}(-\partial_i)\mathrm{mdeg}_{R_N^-} D'_\tau = \sum_\tau m_{\sigma\tau}\, \mathrm{mdeg}_{R_N^-} D_\tau = \frac{1}{A + z_i - z_j} \sum_\tau m_{\sigma\tau}\, \mathrm{mdeg}_{R_N^{\Delta=0}} D_\tau$$

where the latter sums are over only those τ such that D'_τ is not $(GL_2)_i$-invariant (and where the last equality comes from property 3(b) of section 1.5). We finally obtain for $J_\sigma = \mathrm{mdeg}_{R_N^{\Delta=0}} D_\sigma$:[2]

$$-(A + z_i - z_{i+1})\partial_i J_\sigma = \sum_\tau m_{\sigma\tau} J_\tau \tag{5.2}$$

This equation is only valid if $D_\sigma \not\subset \{M_{i,i+1} = 0\}$; however it is easy to see that in the case $D_\sigma \subset \{M_{i,i+1} = 0\}$, it is still satisfied if one conventionally sets $m_{\sigma\sigma} = -2$, $m_{\sigma\tau} = 0$ ($\sigma \neq \tau$). This is slightly strange in that $m_{\sigma\tau}$ is otherwise nonnegative, but this negativity is essentially unavoidable in a sense made precise in a moment.

[2]We could equally well have defined J_σ as $\mathrm{mdeg}_{R_N} D_\sigma$, incurring a factor of A^N. Joseph could not have, since he implicitly works at $A = 0$, in his neglect of the dilation action.

Usually, Eq. (5.2) is rewritten as

$$(-r_i + A\partial_i)\, J_\sigma = -J_\sigma - \sum_\tau m_{\sigma\tau} J_\tau$$

which shows that the $\mathbb{Z}$-span of the $\{J_\tau\}$ is closed under the action of the operators $\{-r_i + A\partial_i\}$. These operators are easily seen to satisfy the S_N Coxeter relations, but what is more, this representation is irreducible, and each irreducible representation of S_N arises from a unique nilpotent orbit.

In particular, these matrices square to 1, which only happens for integer matrices of constant sign in the uninteresting case of permutation matrices (times ± 1) – but those would not generate an irreducible representation. This is the sense mentioned just above in which the flipped sign $m_{\sigma\sigma} < 0$ is unavoidable.

Above, we split the description into two cases according to whether D'_σ was $(GL_2)_i$-invariant or not, but this was only in an attempt to aid understanding rather than mathematically necessary; the equivariant cohomology calculation underlying Lemma 8 does not actually require distinguishing the two cases. That calculation is based on the pushforward of the fundamental class along the map μ, and this pushforward vanishes when the generic fiber is positive-dimensional. Correspondingly, in that case $\mathrm{mdeg}_{M_N(\mathbb{C})} X$ is symmetric in $\{z_i, z_{i+1}\}$, thus annihilated by ∂_i.

5.1.2. *The case* $D = \{M^2 = 0\}$. We include the results in this subsection only to illustrate the formula above, and do not pause to give details of this calculation.

For simplicity let $N = 2n$. Then σ is encoded by a chord diagram on the interval with no crossings, and D_σ is $(GL_2)_i$-invariant iff σ has no arch connecting $i \leftrightarrow i+1$.

Assume now that σ has such an arch. We have already considered the geometry of the hyperplane section in Theorem 6; $D_\sigma \cap \{M_{i,i+1} = 0\}$ has $\overline{B \cdot \tau'_>}$ as a component iff τ' is constructed from σ

- by pulling the $i \leftrightarrow i+1$ arch to touch some other arch, then creating a crossing there, or
- (if there are no crossings, and no arch containing $i \leftrightarrow i+1$) breaking the $i \leftrightarrow i+1$ arch into two vertical lines.

Moreover, these components $D'_\tau = \overline{B \cdot \tau'_>}$ show up with multiplicity 1 (they are generically reduced).

The next step is to sweep each such $\overline{B \cdot \tau'_>}$ using $(GL_2)_i$. If τ' is of the second type listed above, then $\overline{B \cdot \tau'_>}$ is already $(GL_2)_i$-invariant. Hence the k in Lemma 8 is 0, and we can ignore these components. For the τ' of the first type, $(GL_2)_i \cdot \overline{B \cdot \tau'_>} = \overline{B \cdot \tau_>}$, where τ is constructed from τ' by replacing the new (unique) crossing with)(. In this case $k = 1$.

This leads to the following characterization of the τ that arise from σ in this way. Given an arbitrary chord diagram τ without crossings, define $e_i \cdot \tau$ by replacing the two arches

$$i \leftrightarrow \tau(i), \quad i+1 \leftrightarrow \tau(i+1) \qquad \dashrightarrow \qquad i \leftrightarrow i+1, \quad \tau(i) \leftrightarrow \tau(i+1).$$

Then the τ constructed above are those such that $\tau \neq \sigma$, $e_i \cdot \tau = \sigma$. We obtain finally the following recurrence for extended Joseph polynomials:

$$-(A + z_i - z_{i+1})\partial_i J_\sigma = \sum_{\tau \neq \sigma, e_i \cdot \tau = \sigma} J_\tau \qquad \text{if } \sigma(i) = i+1 \tag{5.3}$$

The Hotta construction only applies to orbital varieties, that is to σ noncrossing. However Eq. (5.3) still makes sense for more general σ (i.e. for other B-orbits); clearly the procedure outlined above works equally well if σ has crossings, as long as the arch $(i, i+1)$ is replaced in the preimages τ by a pair of noncrossing arches. This condition on τ, as well as Eq. (5.3) itself, will naturally come out of the more general Brauer construction.

In the course of the derivation of Eq. (5.3), we have obtained the following result, related to sweeping only. First note that in the calculation above the effect of sweeping on our multidegrees with respect to the upper triangle R_N is given by the operator $\tilde{\partial}_i := \widehat{(A+z_i-z_{i+1})}\partial_i \widehat{\frac{1}{A+z_i-z_{i+1}}} = \widehat{\frac{1}{A+z_{i+1}-z_i}}\partial_i \widehat{(A+z_{i+1}-z_i)}$ (where $\widehat{f}$ denotes the multiplication-by-f operator). Then if σ is any chord diagram such that i and $i+1$ are connected to distinct arches that cross each other,

$$-\tilde{\partial}_i J_\sigma = J_{\bar{f}_i^{-1}\cdot\sigma} \qquad \text{if } (i, \sigma(i)) \text{ and } (i+1, \sigma(i+1)) \text{ cross} \tag{5.4}$$

where $\bar{f}_i^{-1} \cdot \sigma$ (the notation will be explained in section 5.5) is by definition the chord diagram in which the crossing of these two arches is replaced with a)(.

In the permutation sector, Eq. (5.4) is nothing but the usual recursion relation satisfied by (double) Schubert polynomials (the extra conjugation of the divided difference operator coming from the factors in Prop. 2).

5.1.3. *An alternate construction: sweeping then cutting.* In the Hotta construction we started with an orbital variety, intersected it with the hyperplane $\{M_{i,i+1} = 0\}$, then swept out each component using $(GL_2)_i$ to get another orbital variety.

There is an alternate geometric construction, in which we start with an orbital variety, and sweep it out using $(GL_2)_i$. This is no longer upper triangular (unless D_σ is $(GL_2)_i$-invariant), but almost; the only lower-triangle entry that may appear[3] is $M_{i+1,i}$. So we intersect with the hyperplane $\{M_{i+1\ i} = 0\}$ to get a schemy union of orbital varieties.

It is easiest to compare the two approaches by looking at the multidegrees. In the cut-then-sweep approach, the geometry computed

$$-\partial_i (A + z_i - z_{i+1}) \mathrm{mdeg}_{M_N(\mathbb{C})} D_\sigma$$

whereas in this sweep-then-cut approach, the geometry computes

$$-(A + z_{i+1} - z_i)\partial_i \, \mathrm{mdeg}_{M_N(\mathbb{C})} D_\sigma.$$

(noting that we use here multidegrees with respect to the full space $M_N(\mathbb{C})$; compare also with Eq. (5.2)).

The operators are very simply related, differing only by 2,

$$-\partial_i \widehat{(A + z_i - z_{i+1})} = -\widehat{(A + z_{i+1} - z_i)}\partial_i \ - 2$$

and as such we don't learn much from this new construction that wasn't already evident in Hotta's cut-then-sweep construction.

[3] This construction seems slightly simpler (or, requiring less art) in the following sense: rather than guessing in advance the condition $\{M_{i,i+1} = 0\}$ that will work well with the sweeping operation, we just sweep and look through E's equations for which conditions to re-impose. "It's easier to ask forgiveness than it is to get permission" – Adm. Grace Hopper

The cut-then-sweep construction has reached full flower in the construction of convolution algebras in geometric representation theory; see e.g. [CG97, Na99]. In that more general setting we do not know an analogue of the sweep-then-cut construction.

5.2. **The actions on the Brauer loop scheme.** We now explore these constructions in the context of the Brauer loop scheme. The main difference is that $(GL_2)_i \cdot E$ violates *two* of E's equations, one linear, one quadratic.

Most of the calculations in this section will be in the context of infinite periodic upper triangular matrices, $R_{\mathbb{Z} \bmod N}$. In this context we will use $(GL_2)_i$ to denote

$$(GL_2)_i := \left\{ \widetilde{M} \in M_{\mathbb{Z} \bmod N}(\mathbb{C}) : \det \begin{bmatrix} M_{i,i} & M_{i,i+1} \\ M_{i+1,i} & M_{i+1,i+1} \end{bmatrix} \neq 0,\ M_{jk} \neq \delta_{jk} \implies j,k \in \{i, i+1\} \bmod N \right\}$$

and B_i its intersection with $R_{\mathbb{Z} \bmod N}$. Note that if $s \in (GL_2)_i$, $\widetilde{M} \in M_{\mathbb{Z} \bmod N}(\mathbb{C})$, then

$$(s\widetilde{M}s^{-1})_{jk} = \widetilde{M}_{jk} \qquad \text{unless } j \text{ or } k \text{ is in } \{i, i+1\} \bmod N.$$

It will be useful to consider the following notation: for $A \in M_{\mathbb{Z} \bmod N}(\mathbb{C})$, let

$$\begin{aligned} A' &= \text{ the } 2 \times 2 \text{ submatrix of } A \text{ using rows } i, i+1 \text{ and columns } i, i+1, \text{ and} \\ A^{\#} &= \text{ the } 2 \times 2 \text{ submatrix of } A \text{ using rows } i, i+1 \text{ and columns } i+N, i+1+N. \end{aligned}$$

It is easy to see that for $s \in (GL_2)_i$, we have $(s \cdot A)' = s' \cdot A'$, $(s \cdot A)^{\#} = s' \cdot A^{\#}$.

We will need some lemmas about slicing and sweeping in this context. The first one produces useful representatives for many arguments, and was already implicitly used in [KZJ07].

Lemma 9. *Let ρ be a partially defined permutation of $\{1, \ldots, N\}$, such that for any i, if $\rho(i)$ and then $\rho(\rho(i))$ are defined, $\rho(\rho(i)) = i$. Pick $s_1, \ldots, s_N$ generic, and define $\widetilde{\rho} \in M_{\mathbb{Z} \bmod N}(\mathbb{C})$ by*

$$\widetilde{\rho}_{jk} = \begin{cases} s_{j'} & \text{if } \pi(j) = k \bmod N,\ k > j > k - N,\ j' = j \bmod N \\ 0 & \text{otherwise.} \end{cases}$$

Construct an involution ρ' from ρ by "promoting each leftward move to a transposition, and making all others fixed points." In formulae,

$$\rho'(j) = \begin{cases} \rho(j) & \text{if } \rho(j) \text{ is defined and } \rho(j) < j \\ j' & \text{if } \rho(j') \text{ is defined, } \rho(j') = j, \text{ and } j' > j \\ j & \text{otherwise} \end{cases}$$

Then

- *$\widetilde{\rho}^2_{i,i+N} \neq 0$ iff $\rho(i), \rho(\rho(i))$ are both defined. If $\widetilde{\rho}^2_{i,i+N} = \widetilde{\rho}^2_{j,j+N} \neq 0$, then $i = j$ or $i = \rho(j)$.*
- *Let M be the $N \times N$ submatrix of $\widetilde{\rho}$ using rows and columns $\{1, \ldots, N\}$. Then $M^2 = 0$ and M is upper triangular, so by Theorem 3, M is B-conjugate to the strict upper triangle of the permutation matrix of a unique involution; this involution is ρ'.*

Proof. First calculate

$$(\widetilde{\rho}^2)_{j,j+N} = \begin{cases} s_j s_{\rho(j)} & \text{if } \rho(j) \text{ and then } \rho(\rho(j)) \text{ are both defined} \\ 0 & \text{otherwise} \end{cases}$$

This, and the genericity of the $\{s_i\}$, imply the first two statements.

That M is strictly upper triangular is tautological. That M^2 is zero follows from the conditions on ρ. Then the matrix M and the strict upper triangle of ρ' have entries in the same places, so are even T-conjugate, thus B-conjugate. $\square$

Lemma 10. *Let $X \subseteq E$, thought of inside $M_{\mathbb{Z} \bmod N}(\mathbb{C})$. So for $A \in X$, the 2×2 matrices $A', A^\#$ are strictly upper triangular and upper triangular, respectively.*

Then the scheme

$$\left((GL_2)_i \cdot X\right) \cap \left\{A : A' \textit{ is strictly upper triangular}, (A^2)^\# \textit{ is upper triangular}\right\}$$

is contained inside E scheme-theoretically, and the set

$$\left((GL_2)_i \cdot X\right) \cap \left\{A : A' \textit{ is upper triangular}, (A^2)^\# \textit{ is upper triangular}\right\}$$

is contained inside E set-theoretically.

If in addition $X \subseteq \{A : A' = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]\}$, then the scheme

$$\left((GL_2)_i \cdot X\right) \cap \left\{A : (A^2)^\# \textit{ is upper triangular}\right\}$$

is contained inside E scheme-theoretically.

Proof. The matrices $s \cdot \widetilde{M} \in (GL_2)_i \cdot X$ come very close to satisfying E's defining equations:

$$\begin{aligned}
(s\widetilde{M}s^{-1})_{jk} &= \widetilde{M}_{jk} && \text{unless } j,k \in \{i, i+1\} \pmod N \\
&= 0 && \text{for } j \geq k, \text{ since } \widetilde{M} \in E \\
(s\widetilde{M}s^{-1})^2_{jk} &= (s\widetilde{M}^2 s^{-1})_{jk} && \\
&= \widetilde{M}^2_{jk} && \text{unless } j,k \in \{i, i+1\} \pmod N \\
&= 0 && \text{for } j + N > k, \text{ since } \widetilde{M} \in E
\end{aligned}$$

The only ones that are satisfied on E and not necessarily on $(GL_2)_i \cdot X$ can be rewritten, for $A \in (GL_2)_i \cdot X$, as the $3+1$ equations

$$A' \text{ is strictly upper triangular}, (A^2)^\# \text{ is upper triangular}.$$

The first three equations are implied *set-theoretically* by the single equation $A_{i+1,i} = 0$, since A' is nilpotent (being conjugate to $\widetilde{M}'$, which is strictly upper triangular for $\widetilde{M} \in E$).

So for any subscheme $X \subseteq E$, to intersect $(GL_2)_i \cdot X$ with E it suffices set-theoretically to intersect with the hypersurfaces defined by

$$\widetilde{M}_{i+1,i} = 0, \qquad (\widetilde{M}^2)_{\,i+1,i+N} = 0,$$

whose multidegrees are $A + z_{i+1} - z_i$, $2\mathrm{A} + z_{i+1} - z_{i+N} = \mathrm{A} + \mathrm{B} + z_{i+1} - z_i$ respectively.

Finally, if $X \subseteq \left\{\widetilde{M} : \widetilde{M}' = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]\right\}$, then $(s \cdot \widetilde{M})' = s' \cdot \widetilde{M}' = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ so the first three equations are automatic. $\square$

The calculation above was significantly more complicated in [KZJ07], where we didn't make proper use of $R_{\mathbb{Z} \bmod N}$.

5.2.1. *Sweeping then cutting: the* f_i *action.* We now recall the results of Section 4.2 of [KZJ07], adapted to the present discussion. We have been able to streamline Proposition 6 (the technical heart) of that paper enough that we include here a complete proof, repeating some parts of the one from [KZJ07].

Begin with a chord diagram π with no "little arch" connecting i to $i+1$, and the corresponding Brauer loop variety $E_\pi \subseteq \mathcal{M}_N$. Sweep E_π out using the $(GL_2)_i$ action. Lifting its elements to $R_{\mathbb{Z} \bmod N}$, an element lying over the sweep $(GL_2)_i \cdot E_\pi$ looks like

$$s\widetilde{M}s^{-1}, \qquad \text{where} \quad s \in (GL_2)_i, \quad \widetilde{M}^2 \in \langle S^N \rangle, \quad (M^2)_{i,i+N} = (M^2)_{\pi(i),\pi(i)+N}.$$

Calculating the degree. We first show the map $(GL_2)_i \times^{B_i} E_\pi \to (GL_2)_i \cdot E_\pi$ has degree 1. We must select an $\widetilde{M} \in (GL_2)_i \cdot E_\pi$ and compute the fiber $\left\{[s, \widetilde{M}'] : s \cdot \widetilde{M}' = \widetilde{M}\right\}$ lying over it. By Theorem 1, for general elements $\widetilde{M}$ lying over E_π we know $\widetilde{M}^2_{i,i+N}, \widetilde{M}^2_{i+1,i+1+N}$ are different. Fix such an $\widetilde{M} = 1 \cdot \widetilde{M} \in (GL_2)_i \cdot E_\pi$. Then $(\widetilde{M}^2)^\#$ is upper triangular with distinct diagonal entries, and

$$\widetilde{M}^2 = s \cdot \widetilde{M}'^2 \quad \Longrightarrow \quad (\widetilde{M}^2)^\# = \left(s \cdot \widetilde{M}'^2\right)^\# = s^\# \cdot (\widetilde{M}'^2)^\#$$

where $(\widetilde{M}'^2)^\#$ is also upper triangular. By considering the eigenvalues of the 2×2 matrices $(\widetilde{M}^2)^\#, (\widetilde{M}'^2)^\#$, we see their diagonals must either agree or be reversed. Since $\widetilde{M}' \in E_\pi$, by Theorem 1 their diagonals must agree. Since $s^\#$ conjugates a 2×2 upper triangular matrix with distinct eigenvalues to another with the same diagonal, $s^\#$ too must be upper triangular, so $s \in B_i$. This shows that the fiber $\left\{[s, \widetilde{M}'] : s \cdot \widetilde{M}' = \widetilde{M}\right\}$ over $\widetilde{M}$ is a point, hence the degree of the map is 1. In particular, E_π is not $(GL_2)_i$-invariant.

The necessary extra equation. By this non-invariance, $\dim((GL_2)_i \cdot E_\pi) > \dim E_\pi = \dim E$. Hence the irreducible variety $(GL_2)_i \cdot E_\pi$ does not lie in E, and by the last conclusion in Lemma 10 the only equation we need re-impose is $(\widetilde{M}^2)_{i+1,i+N} = 0$. Let F denote the intersection of $(GL_2)_i \cdot E_\pi$ and $\left\{\widetilde{M} : (\widetilde{M}^2)_{i+1,i+N} = 0\right\}$.

Since F is a Cartier divisor in an irreducible variety, its geometric components are all of codimension 1 in $(GL_2)_i \cdot E_\pi$, hence of the same dimension as E. So as a set, F is a union of some of the top-dimensional components of E. Since $F \subseteq E$ (by Lemma 10), and E is generically reduced along its top-dimensional components, F is too.

Hence F is a union of some $\{E_\rho\}$, up to embedded components, which don't affect multidegree calculations.

The geometric components of F*.* It is easy to see that E_π is a component:

$$E_\pi = 1 \cdot E_\pi \subseteq ((GL_2)_i \cdot E_\pi) \cap \left\{(\widetilde{M}^2)_{i+1,i+N} = 0\right\} =: F.$$

To determine which other $E_{\pi'}$ are in F, we use Theorem 1, which characterizes the components using the functions $\{\widetilde{M}^2_{j,j+N}\}$.

$$(s\widetilde{M}s^{-1})^2_{j,j+N} = (s\widetilde{M}^2 s^{-1})_{j,j+N} \qquad = \widetilde{M}^2_{j,j+N} \text{ unless } j \in \{i, i+1\} \bmod N$$

hence

$$(s\widetilde{M}s^{-1})^2_{j,j+N} = (s\widetilde{M}s^{-1})^2_{\pi(j),\pi(j)+N} \qquad \text{unless } j \text{ or } \pi(j) \in \{i, i+1\} \bmod N,$$

so by Theorem 1 (and assuming now, without loss of generality, that $1 \le i, j \le N$)

$$E_{\pi'} \subseteq F \qquad \Longrightarrow \qquad \pi'(j) = \pi(j) \quad \text{for} \quad j \notin \{i, i+1, \pi(i), \pi(i+1)\}.$$

(This calculation, too, was more complicated in [KZJ07] for not using $R_{\mathbb{Z} \bmod N}$.)

There are three ways to link up $\{i, i+1, \pi(i), \pi(i+1)\}$, namely π, $f_i \cdot \pi$, and $e_i \cdot \pi$, but the third is ruled out by $(s\widetilde{M}s^{-1})_{i,i+1} = 0$. Hence

$$E_{\pi'} \subseteq F \qquad \Longrightarrow \qquad \pi' = \pi \ \text{ or } \ \pi' = f_i \cdot \pi.$$

This leaves two possibilities for the *set* F:

$$F \quad \text{is either} \quad E_\pi \quad \text{or} \quad E_\pi \cup E_{f_i \cdot \pi}.$$

To rule out $F = E_\pi$, we exhibit an element of $F \setminus E_\pi$. Let $\widetilde{M} \in M_{\mathbb{Z} \bmod N}(\mathbb{C})$ be the element constructed in Lemma 9. In particular $\widetilde{M} \in E_\pi$, and $\widetilde{M} \notin E_\rho$ for any $\rho \neq \pi$. Let $s \in (GL_2)_i$ be the element with $s' = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. Then $(s \cdot \widetilde{M})^2 = s \cdot \widetilde{M}^2$ is supported on the same superdiagonal, but with the $(i, i+N), (i+1, i+N+1)$ elements exchanged. So $s \cdot \widetilde{M} \in E_{f_i \cdot \pi}$, and not in E_π, as intended.

(It does not follow from this that $F = E_\pi \cup E_{f_i \cdot \pi}$ as a scheme, *even if* one also believes that E is reduced: F may still have embedded components along the intersections of E_π, $E_{f_i \cdot \pi}$ with other components of E.)

Multidegrees. That completes our geometric analysis of F, and we move on now to the consequence for multidegrees: $\operatorname{mdeg} F = \operatorname{mdeg} E_\pi + \operatorname{mdeg} E_{f_i \cdot \pi}$. By Lemma 8

$$\operatorname{mdeg} F = (A + B + z_{i+1} - z_i)(-\tilde{\partial}_i \operatorname{mdeg} E_\pi) \tag{5.5}$$

(the factor k required in Lemma 8 is 1, as we showed at the beginning) where the factor $A + B + z_{i+1} - z_i$ is the weight of the equation $(M^2)_{i+1,i+N} = 0$, and where $\tilde{\partial}_i := \frac{1}{\widehat{A+z_{i+1}-z_i}} \partial_i (\widehat{A + z_{i+1} - z_i})$, the extra conjugation of ∂_i compared to Lemma 8 coming from the fact that we consider multidegrees with respect to upper triangular matrices.

We conclude that

$$-(A + B + z_{i+1} - z_i)\tilde{\partial}_i \Psi_\pi = \Psi_\pi + \Psi_{f_i \cdot \pi}. \tag{5.6}$$

which is Eq. (4.19) with $\Psi_\pi = \operatorname{mdeg} E_\pi$.

(The possibility $F = E_\pi$ we ruled out above can also be excluded using multidegrees, which was how we did it in [KZJ07]. Assuming $\operatorname{mdeg} F = \operatorname{mdeg} E_\pi$ leads quickly to the equation $\tilde{\partial}_i \operatorname{mdeg} E_\pi = 2\tilde{\partial}_i \operatorname{mdeg} E_\pi$. But $\tilde{\partial}_i \operatorname{mdeg} E_\pi$ is the multidegree of a subscheme of a representation whose T-weights all live in a half-space, so cannot be zero.)

An interesting difference between this construction and the one in section 5.1.3 is the use of a *quadratic* equation $(M^2)_{i+1,i+N} = 0$ rather than a linear equation $M_{i+1,i} = 0$.

5.3. **Connection with the qKZ equation.** We can now formulate at last the main result of this paper:

Theorem 9. *The following two vector-valued polynomials* $\Psi = \sum_\pi \Psi_\pi \pi \in V[A, \epsilon, z_1, \ldots, z_N]$ *coincide:*

(i) *the solution of Eqs. (4.6–4.7) given by Theorem 8 with its normalization fixed by Eq. (4.14), and*

(ii) *the vector of multidegrees* $\Psi_\pi = \mathrm{mdeg}_{\mathcal{M}_N^{\Delta=0}} E_\pi$ *of the irreducible components of the Brauer loop scheme (with the identification* $B := A - \epsilon$*).*

Proof. The multidegrees $\Psi_\pi = \mathrm{mdeg}_{\mathcal{M}_N^{\Delta=0}} E_\pi$ are by definition homogeneous polynomials in $\mathbb{Z}[A, \epsilon, z_1, \ldots, z_N]$, of degree the codimension of the E_π, which is shown in Theorem 3 of [KZJ07] to be $2n(n-1)$. Furthermore, they satisfy Eq. (5.6), which is identical to Eq. (4.9). Thus, they fulfill all the hypotheses of Lemma 6 to ensure uniqueness of the solution of Eqs. (4.6–4.7) up to normalization. The latter is fixed by considering the base case $\pi_0(i) = i + n$: as stated in the proof of Prop. 5 of [KZJ07], E_{π_0} is a linear variety given by the equations $M_{ij} = 0$, $i = 1, \ldots, N$, $j = i+1, \ldots, i+n-1$, hence its multidegree matches Eq. (4.14). □

Corollary 1. *The vector of multidegrees* $\Psi_\pi = \mathrm{mdeg}_{\mathcal{M}_N^{\Delta=0}} E_\pi$ *of the Brauer loop varieties satisfies Equation (4.20).*

In the next subsection we give a geometric interpretation of this equation.

5.4. **Geometry of the e_i action.** In this section we give a new geometric construction, promised after Corollary 4 of [KZJ07], to handle the case $\pi(i) = i + 1$.

- cut E_π with the hyperplane $\{M_{i,i+1} = 0\}$, producing F_1;
- throw away the $(GL_2)_i$-invariant components, and sweep out what remains with $(GL_2)_i$, producing F_2; then
- cut F_2 with $\{(M^2)_{i+1,i+N} = 0\}$, giving F_3.

We will identify F_3, not as a union of Brauer loop varieties, but the intersection of that with a hypersurface. Taking multidegrees, we will reproduce Equation (4.20). This is not quite a geometric *proof*, as we will only follow the geometry close enough to get an upper bound, and invoke Equation (4.20) to show the bound is tight.

The algebraic details of this proof, in Section 5.4.3, only make reference to F_1 and not explicitly F_2, F_3. However they only involve $\mathrm{mdeg}\, F_1$ in the form $(-\partial_i)\mathrm{mdeg}\, F_1$, which is a sign that the computation is "really" about F_2, not F_1. And since $F_2 \not\subseteq E$, it seems more natural to follow it to $F_3 \subseteq E$ (set containment). In Section 5.4.6 we follow this to determine F_3, up to lower-dimensional embedded components.

Even to determine the dimensions of F_1, F_2, F_3 will require work, as each cutting operation may or may not decrease the dimension by 1. It will turn out they do, so F_1, F_3 are codimension 1 in E, and F_2 is the same dimension as E (but is not contained in it). To analyze them, we will need to make use of a certain family of codimension 1 subvarieties of E.

5.4.1. *The subvarieties* $\{X_\rho\}$. Consider the link patterns $\rho \neq \pi$ such that $e_i \cdot \rho = \pi$, and *provisionally* define

$$X_\rho := E_\rho \cap \left\{ M \in E : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N} \right\}. \tag{5.7}$$

Since X_ρ is a Cartier divisor in the variety E_ρ, and the equation is nontrivial (it is not satisfied on the point $\widetilde{\rho}$ from Lemma 9), the set X_ρ has pure codimension 1 in E.

Each such ρ agrees with π away from $i, i+1, \pi(i), \pi(i+1)$. There are three ways to hook up these four spots, namely π, ρ, and $f_i \cdot \rho$, and as such the set of these ρ comes in pairs.

We can pick a representative from each pair in a natural way: let

$$R(\pi, i) = \{\rho \neq \pi \; : \; e_i \cdot \rho = \pi, \text{ and the chords emanating from } i, i+1 \text{ cross each other}\}.$$

Proposition 9. *Let $\rho \in R(\pi, i)$. Then X_ρ is irreducible, whereas $X_{f_i \cdot \rho}$ has two geometric components, one of which is (the reduction of) X_ρ; call the other Y_p. Both schemes are generically reduced.*

As this proposition already suggests, it will be much more convenient hereafter to *redefine* X_ρ as the reduction of the X_ρ above. (In fact, we conjecture that E_ρ is normal, which would imply that X_ρ is reduced.) This of course doesn't change X_ρ's multidegree, which is what most interests us.

Proof. By cyclic invariance we can assume that $i = 1$. We shall use in this proof the (R, L) decomposition of Section 1.1, as well as the equations of Thm. 2. Recall from [KZJ07] that the projection $(R, L) \mapsto R$ allows one, using Thm. 3, to decompose E as a disjoint union of preimages F_α of orbits $B \cdot \alpha_<$, where α is an involution. Furthermore this makes F_α a *vector bundle* over $B \cdot \alpha_<$, implying in particular that it is irreducible.

Let $\rho \in R(\pi, i)$, α be an involution and consider $X_\rho \cap F_\alpha$. We shall show that if $\alpha \neq \rho$, the result is empty or of dimension strictly less than $\dim E - 1 = 2n^2 - 1$.

F_α is a vector bundle over $B \cdot \alpha_<$; its dimension was computed in the proof of Thm. 3 of [KZJ07] and found to be $2n^2 - \frac{1}{2}\#$fixed-points (fixed points being half-lines in the language of Section 2). Let us now impose additional equations (from those in the proof of Thm. 3 of [KZJ07]) on the fiber by intersecting with X_ρ. We choose the particular fiber for which the upper triangular part is $R = \alpha_<$.

Requiring any element $M \in F_\alpha$ to satisfy the rank equations (3) of Thm. 2 for E_ρ implies that $\alpha_< \leq \rho_<$ (with respect to the order from Section 2.1), and in particular that $\alpha(i) \neq i+1$ (this is where we use that $i = 1$). Now suppose that i and $i+1$ are not both fixed points of α. Then the equation $(M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}$ or more explicitly $L_{i\leftrightarrow\alpha(i)} = L_{i+1\leftrightarrow\alpha(i+1)}$ ($a \leftrightarrow b := \max(a, b), \min(a, b)$) if neither are fixed points, or $L_{i\leftrightarrow\alpha(i)} = 0$ or $L_{i+1\leftrightarrow\alpha(i+1)} = 0$ if one of them is, is an additional linear equation *on the fiber* which reduces its dimension by 1 and keeps it reduced. If α is not a link pattern, we are already done since the resulting dimension is less than $2n^2 - 1 = \dim E - 1$. So we assume in what follows that either α is a link pattern, or $\alpha(i) = i$ and $\alpha(i+1) = i+1$ (and there are no other fixed points, by dimensionality). In both cases we are already in dimension $2n^2 - 1$.

Now we make use of $\alpha \neq \rho$, forcing $\alpha_< < \rho_<$. Note that $\pi_< \geq \rho_<$ and $(f_i \cdot \rho)_< \geq \rho_<$, so that $\alpha \notin \{\pi, \rho, f_i \cdot \rho\}$. Hence α cannot be equal to ρ outside $\{i, i+1, \rho(i), \rho(i+1)\}$ (paying attention to the special case where i and $i+1$ are fixed points); there is at least one more pair $(j, \rho(j))$ distinct from these four elements such that $\alpha(j) \neq \rho(j)$. It is now easy to check that equation (2) of Thm. 2 for E_ρ (with i replaced with j) produces one more linear equation on the fiber which reduces further the dimension by 1. We conclude that $X_\rho \cap F_\alpha$ is a subset of a vector bundle over $B \cdot \alpha_<$ which is of dimension $\dim E - 2$.

Finally, if $\alpha = \rho$, we have

$$X_\rho \cap F_\rho = F_\rho \cap \left\{M \in E : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}\right\}$$

and conclude immediately from the equations above that $X_\rho \cap F_\rho$, just like F_ρ, is a vector bundle over $B \cdot \pi_<$ and therefore irreducible. Now look at the decomposition

$$X_\rho = \coprod_\alpha (X_\rho \cap F_\alpha) = (X_\rho \cap F_\rho) \cup \coprod_{\alpha \neq \rho} (X_\rho \cap F_\alpha):$$

the first piece $X_\rho \cap F_\rho$ is reduced and irreducible, the other pieces $X_\rho \cap F_{\alpha\neq\rho}$ are of lower dimension, and X_ρ is set-theoretically equidimensional. Putting these facts together, we see that X_ρ is irreducible, generically reduced, and contains $X_\rho \cap F_\rho$ as a dense subset.

Now we take up $X_{f_i\cdot\rho} \cap F_\alpha$, whose analysis is similar. Note that the inequality $\alpha_< \leq (f_i \cdot \rho)_<$ has two interesting solutions, namely $\alpha = \rho$ or $\alpha = f_i \cdot \rho$. We leave the reader to check that the same dimension computation as above shows that if $\alpha \notin \{\rho, f_i \cdot \rho\}$, then the dimension of $X_{f_i\cdot\rho} \cap F_\alpha$ is strictly less than $\dim E - 1$. Again, the condition of lying in $X_{f_i\cdot\rho}$ is a linear condition on the fibers of this vector bundle, so the intersections are reduced and irreducible.

If $\alpha = f_i \cdot \rho$,

$$X_{f_i\cdot\rho} \cap F_{f_i\cdot\rho} = F_{f_i\cdot\rho} \cap \left\{M \in E : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}\right\}$$

so that $Y_\rho := \overline{X_{f_i\cdot\rho} \cap F_{f_i\cdot\rho}}$ is a geometric component of $X_{f_i\cdot\rho}$ of dimension $\dim E - 1$.

If $\alpha = \rho$, we have

$$X_{f_i\cdot\rho} \cap F_\rho = E_{f_i\cdot\rho} \cap (X_\rho \cap F_\rho) \subset X_\rho$$

so by dimensionality, $\overline{X_{f_i\cdot\rho} \cap F_\rho} = X_\rho$. □

5.4.2. F_1. First we note that the equation $M_{i,i+1} = 0$ does not hold on the point $\widetilde{\pi} \in E_\pi$ constructed in Lemma 9. So F_1 is a Cartier divisor in the irreducible variety E_π, hence equidimensional of dimension $\dim E - 1$, and by axiom (3c) of multidegrees,

$$\operatorname{mdeg} F_1 = (A + z_i - z_{i+1}) \operatorname{mdeg} E_\pi$$

where the linear factor is the multidegree of the hyperplane $\{M_{i,i+1} = 0\}$.

Since that hyperplane is B-invariant inside $R_{\mathbb{Z} \bmod N}$, so is F_1 and each of its components.

Lemma 11. *Let* $\rho \in R(\pi, i)$. *Then* $X_\rho \subseteq F_1$.

Proof. A slightly more explicit description of X_ρ, found in appendix B, implies (Lemma 16) that $X_\rho \subset E_\pi$. Since $\rho(i) \neq i+1$, $E_\rho \subseteq \{M_{i,i+1} = 0\}$. So $X_\rho \subseteq E_\rho \cap E_\pi \subseteq E_\pi \cap \{M_{i,i+1} = 0\} = F_1$. □

So the X_ρ are (by construction distinct) irreducible components of F_1. (They will turn out to be all the $(GL_2)_i$-non-invariant components.)

5.4.3. *The multidegree of this lower bound* $\bigcup_{\rho \in R(\pi,i)} X_\rho$. It will be convenient to work in and compute multidegrees relative to a $(GL_2)_i$-invariant space P_i, one dimension larger than $R_{\mathbb{Z} \bmod N}$ in that it includes the possibility that the $\widetilde{M}_{i+1,i}$ entry just below the diagonal may be nonzero. Then for $X \subseteq R_{\mathbb{Z} \bmod N}$ and T-invariant,

$$\operatorname{mdeg}_{P_i} X = (A + z_{i+1} - z_i) \operatorname{mdeg} X.$$

In particular, the multidegree of our lower bound on F_1 is

$$\begin{aligned}
\mathrm{mdeg}_{P_i} \bigcup_{\rho\in R(\pi,i)} X_\rho &= \sum_{\rho\in R(\pi,i)} \mathrm{mdeg}_{P_i} X_\rho \\
&= \sum_{\rho\in R(\pi,i)} \mathrm{mdeg}_{P_i} \left(E_\rho \cap \left\{M \in P_i : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}\right\}\right) \\
&= \sum_{\rho\in R(\pi,i)} (A+B)\mathrm{mdeg}_{P_i} E_\rho \\
&= (A+B) \sum_{\rho\in R(\pi,i)} \mathrm{mdeg}_{P_i} E_\rho
\end{aligned}$$

The right side of Equation (4.20), times $A + z_{i+1} - z_i$, is related to this lower bound:

$$\begin{aligned}
& (A + z_{i+1} - z_i)(A+B) \sum_{\rho\neq\pi, e_i\cdot\rho=\pi} \Psi_\rho \\
= \;& (A+B) \sum_{\rho\neq\pi, e_i\cdot\rho=\pi} (A + z_{i+1} - z_i)\Psi_\rho \\
= \;& (A+B) \sum_{\rho\neq\pi, e_i\cdot\rho=\pi} \mathrm{mdeg}_{P_i} E_\rho \\
= \;& (A+B) \sum_{\rho\in R(\pi,i)} \left(\mathrm{mdeg}_{P_i} E_\rho + \mathrm{mdeg}_{P_i} E_{f_i\cdot\rho}\right) \\
= \;& (A+B) \sum_{\rho\in R(\pi,i)} (A + B + z_{i+1} - z_i)(-\partial_i)\mathrm{mdeg}_{P_i} E_\rho \qquad \text{by equation (5.5)} \\
= \;& (A + B + z_{i+1} - z_i)(-\partial_i)(A+B) \sum_{\rho\in R(\pi,i)} \mathrm{mdeg}_{P_i} E_\rho \\
= \;& (A + B + z_{i+1} - z_i)(-\partial_i)\, \mathrm{mdeg}_{P_i} \bigcup_{\rho\in R(\pi,i)} X_\rho
\end{aligned}$$

At this point it is tempting to use our scheme-theoretic bound on F_1 to infer

$$\text{“}\mathrm{mdeg}_{P_i} \bigcup_{\rho\in R(\pi,i)} X_\rho \;\leq\; \mathrm{mdeg}_{P_i} F_1'' \qquad \text{and} \tag{5.8}$$

$$\text{“}(A + B + z_{i+1} - z_i)(-\partial_i)\, \mathrm{mdeg}_{P_i} \bigcup_{\rho\in R(\pi,i)} X_\rho \;\leq\; (A + B + z_{i+1} - z_i)(-\partial_i)\, \mathrm{mdeg}_{P_i} F_1'' \tag{5.9}$$

on multidegrees; we make this precise in the next section. For now, we analyze

$$\begin{aligned}
& (A + B + z_{i+1} - z_i)(-\partial_i)\, \mathrm{mdeg}_{P_i} F_1 \\
= \;& (A + B + z_{i+1} - z_i)(-\partial_i)\, \mathrm{mdeg}_{P_i} (E_\pi \cap \{M_{i,i+1} = 0\}) \\
= \;& (A + B + z_{i+1} - z_i)(-\partial_i)\, (A + z_i - z_{i+1})\mathrm{mdeg}_{P_i} E_\pi \\
= \;& (A + B + z_{i+1} - z_i)(-\partial_i)\, (A + z_i - z_{i+1})(A + z_{i+1} - z_i)\mathrm{mdeg}\, E_\pi \\
= \;& (A + B + z_{i+1} - z_i)(A + z_i - z_{i+1})(A + z_{i+1} - z_i)(-\partial_i)\, \mathrm{mdeg}\, E_\pi \\
= \;& (A + B + z_{i+1} - z_i)(A + z_i - z_{i+1})(A + z_{i+1} - z_i)(-\partial_i)\, \Psi_\pi
\end{aligned}$$

and obtain $A + z_{i+1} - z_i$ times the left side of Equation (4.20).

5.4.4. *Inequalities on multidegrees.* It remains to develop the technology to interpret and prove the inequalities (5.8) and (5.9).

Lemma 12. *Let W be a representation of a torus T. Assume that all weights of W lie in a proper cone in T^*. Then we can define a proper cone in $\mathrm{Sym} T^*$, consisting of $\mathbb{N}$-combinations of monomials in the weights in W. We can define a partial order $\leq$ on $\mathrm{Sym} T^*$ where $f \leq g$ if $g - f$ lies in this cone. Then for any nonempty T-invariant scheme $X \subseteq W$, $\mathrm{mdeg}_W X > 0$.*

Let $X, Y \subseteq W$ be two T-invariant subschemes of the same dimension, with $X \subseteq Y$. Then $\mathrm{mdeg}_W X \leq \mathrm{mdeg}_W Y$. This inequality is an equality iff $Y \setminus X$ is of lower dimension.

If Y is generically reduced along its top-dimensional components, then X is also. Assume now that $\mathrm{mdeg}_W X = \mathrm{mdeg}_W Y$. If Y is reduced and equidimensional, then $X = Y$. (This last conclusion also appears in [KM05, Lemma 1.7.5] *and, stated less generally, in* [Mar02]*.)*

Proof. The condition that the weights lie in a proper cone is equivalent to the existence of a linear functional $T^* \to \mathbb{R}$ taking each weight in W to a strictly positive number. That functional extends to a ring homomorphism $\mathrm{Sym} T^* \to \mathbb{R}$, in which $\mathbb{N}$-combinations of monomials in these weights (other than the null combination) also go to strictly positive numbers.

With this one can show that this cone only intersects its negative in $\{0\}$, which says that $f \leq g \leq f \implies f = g$. The other properties of a partial order are obvious. The fact that $\mathrm{mdeg}_W X$ is a nontrivial sum of products of monomials in W is easy to prove by induction from the axiomatic definition of multidegrees.

Let $\{Y_i\}$ be the top-dimensional components of Y, with multiplicities $\{m_i(Y)\}$, so $\mathrm{mdeg}_H Y = \sum_i m_i(Y) \mathrm{mdeg}_H Y_i$. Since $X \subseteq Y$ and they have the same dimension, each $\dim X$-dimensional component of X is a component of Y hence some Y_i; let $m_i(X)$ be the multiplicity of Y_i in X. Then $X \subseteq Y$ implies $m_i(X) \leq m_i(Y)$, so $\mathrm{mdeg}_H Y - \mathrm{mdeg}_H X = \sum_i (m_i(Y) - m_i(X)) \mathrm{mdeg}_H Y_i$. Since each $\mathrm{mdeg}_H Y_i > 0$, this sum is ≥ 0 with equality iff $m_i(Y) = m_i(X)$ for each i, iff X contains all of Y's top-dimensional components and is generically equal to Y on each one. That is equivalent to $Y \setminus X$ being of lower dimension.

Since $Y \supseteq X$ of the same dimension, if Y is generically reduced along its top-dimensional components, then X is also. If Y is reduced and equidimensional, then Y is the union of its geometric components. If $\mathrm{mdeg}_H X = \mathrm{mdeg}_H Y$, then all of these components appear in X; hence $X \supseteq Y$, but we were given $X \subseteq Y$. $\square$

This combines nicely with Lemma 8:

Lemma 13. *Let $V \leq M_N(\mathbb{C})$ be a subspace invariant under B and $(GL_2)_i$. Let X be an equidimensional scheme in V invariant under B and rescaling, with multidegree $\mathrm{mdeg}_V X$. If each geometric component of X is $(GL_2)_i$-invariant, then $\partial_i \,\mathrm{mdeg}\, X = 0$. Otherwise,*

$$(-\partial_i)\,\mathrm{mdeg}\, X \;\geq\; \mathrm{mdeg}((GL_2)_i \cdot X),$$

where $(GL_2)_i \cdot X$ is given the reduced scheme structure. This is an equality iff the degrees $\{k_j\}$ from lemma 8 are 0 or 1, and each $(GL_2)_i$-non-invariant component X_j is generically reduced.

In particular, $\partial_i \,\mathrm{mdeg}\, X = 0$ implies each geometric component of X is $(GL_2)_i$-invariant.

Proof. Break X into its components X_j, with multiplicities $m_j > 0$, and apply Lemma 8,

$$\begin{aligned}(-\partial_i)\operatorname{mdeg} X &= (-\partial_i)\sum_{X_j} m_j \operatorname{mdeg} X_j = \sum_{X_j} m_j(-\partial_i)\operatorname{mdeg} X_j \\ &= \sum_{X_j} m_j k_j \operatorname{mdeg}((GL_2)_i \cdot X_j) = \sum_{X_j:\ k_j\neq 0} m_j k_j \operatorname{mdeg}((GL_2)_i \cdot X_j) \\ &\geq \sum_{X_j:\ k_j\neq 0} \operatorname{mdeg}((GL_2)_i \cdot X_j),\end{aligned}$$

using also the fact that $\operatorname{mdeg}((GL_2)_i \cdot X_j) > 0$. For this to be an equality, each $k_j \neq 0$ must be equal to 1, and the corresponding m_j must be 1. For the left side to be zero, all k_j must be zero, which by Lemma 8 says that the components are $(GL_2)_i$-invariant. $\square$

5.4.5. *Wrapping up.* Using Lemma 13, we conclude the inequality (5.9), and from the analysis in Section 5.4.3 obtain Equation (4.20) as an inequality. Since we knew by other means that Equation (4.20) holds on the nose, the inequality (5.9) is also an equality.

Hence by Lemma 13,

Theorem 10. *Let* F_1 *be as defined at the beginning of Section 5.4, and* $\{X_\rho,\ \rho \in R(\pi, i)\}$ *the components of* F_1 *defined in Section 5.4.1. Then*

- F_1 *is generically reduced along each* X_ρ*.*
- *All other components of* F_1 *are* $(GL_2)_i$*-invariant.*
- *The map from* $(GL_2)_i \times^{B_i} X_\rho \twoheadrightarrow (GL_2)_i \cdot X_\rho$ *is birational.*

5.4.6. *Epilogue:* F_3. A priori, some of F_2's components could be contained in the hypersurface $\left\{M : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}\right\}$ and some not, in which case F_3 would have components of dimensions $\dim E$ and $\dim E - 1$. In fact only the latter occur:

Lemma 14. *Separate* E *into* $E_{i\leftrightarrow i+1} \cup E_{i\not\leftrightarrow i+1}$*, where*

$$E_{i\leftrightarrow i+1} = \bigcup_{\rho:\ \rho(i)=i+1} E_\rho, \qquad E_{i\not\leftrightarrow i+1} = \bigcup_{\rho:\ \rho(i)\neq i+1} E_\rho.$$

Then the set $F_3 := F_2 \cap \left\{\widetilde{M} : \left(\widetilde{M}^2\right)_{i+1,i+N} = 0\right\}$ *consists of components of*

$$(E_{i\leftrightarrow i+1} \cap \{M : M_{i\ i+1} = 0\}) \cup \left(E_{i\not\leftrightarrow i+1} \cap \{M : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}\}\right)$$

which is equidimensional of codimension 1 *in* E*.*

Proof. Note that $(E_{i\leftrightarrow i+1} \cap \{M : M_{i\ i+1} = 0\})$ and $\left(E_{i\not\leftrightarrow i+1} \cap \{M : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}\}\right)$ are each codimension 1 in E, because for each component E_ρ (of one or the other) the stated equation on M does not hold at the point $\widetilde{\rho}$ constructed in lemma 9.

However, $E_{i\leftrightarrow i+1} \subseteq \{M : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}\}$ and $E_{i\not\leftrightarrow i+1} \subseteq \{M : M_{i\ i+1} = 0\}$, so the set purportedly containing F_3 could equally well be described as

$$E \cap \left\{M : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}\right\} \cap \{M : M_{i\ i+1} = 0\}.$$

Since E_π lies in the intersection of the first two terms, F_1 lies in the intersection of all three. Lemma 10 tells us that $F_2 \subseteq E$ as a set (and hence $F_3 \subseteq E$ as well). To see the other two

conditions, we look at A', $(A^2)^\#$ for $A \in F_{0\ldots 3}$:

	F_0	F_1	F_2	F_3
A'	$\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
$(A^2)^\#$	$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$	$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$	X	$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

where X is conjugate to $\left[\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right]$, and hence has repeated eigenvalues. When we impose once more that its lower left entry is zero (cutting F_2 down to F_3), we recover the condition $\{M : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}\}$, as pictured in the table. This elementwise calculation proves the conditions claimed of the set F_3.

Finally, since F_3 is the same dimension as the set

$$E \cap \{M : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}\} \cap \{M : M_{i\ i+1} = 0\}$$

that contains it, F_3's underlying set is a union of components of this larger set. $\square$

It is not hard to tighten this bound to $F_1 \cup \left(\bigcup_{\rho:\ \rho\neq\pi, e_i\cdot\rho=\pi} E_\rho\right) \cap \{M : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}\}$, but the weaker result is already enough to derive

Corollary 2.

$$\mathrm{mdeg}_{P_i} F_3 = (A + B + z_{i+1} - z_i)\ \mathrm{mdeg}_{P_i} F_2 = (A+B) \sum_{\rho\neq\pi, e_i\cdot\rho=\pi} \mathrm{mdeg}_{P_i} E_\rho.$$

Proof. Break F_2 into its components X_ρ and intersect with $\{M : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}\}$. We have just shown that no component is contained in this hypersurface, so applying properties (2), (3c), (2) we obtain the first equality. The second equality was obtained in Section 5.4.4. $\square$

This last is also the multidegree of $\left(\bigcup_{\rho:\ \rho\neq\pi, e_i\cdot\rho=\pi} E_\rho\right) \cap \left\{M : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}\right\}$, a union of some of the components of the scheme in Lemma 14. Our final goal is to show that F_3 is indeed equal to that union, and more specifically,

$$((GL_2)_i \cdot X_\rho) \cap \left\{M : (M^2)_{i+1,i+N} = 0\right\} = X_{f_i\cdot\rho} = X_\rho \cup \text{ another component } Y_\rho \tag{5.10}$$

for all $\rho \in R(\pi, i)$. X_ρ is contained in $(GL_2)_i \cdot X_\rho$, and since the right-hand side is contained in E, it is contained in $\left\{M : (M^2)_{i+1,i+N} = 0\right\}$. So it remains to prove

$$(GL_2)_i \cdot X_\rho \quad \supseteq \quad Y_\rho$$

We use the description of a dense subset inside Y_ρ given in appendix B. More precisely, we exhibit an element of $GL(2)_i$ and an element of X_ρ which produce a representative of each orbit of Lemma 17. Consider such an orbit representative $M = \underline{f_i \cdot \rho}\, t$, that is in the rows and columns of interest,

$$\begin{array}{c|ccccccc} & i & i{+}1 & \rho(i) & \rho(i{+}1) & i{+}N & i{+}N{+}1 & \rho(i){+}N \\ \hline i & 0 & 0 & 0 & t_i & \cdots & & \\ i{+}1 & & 0 & t_{i+1} & 0 & 0 & \cdots & \\ \rho(i) & & & 0 & 0 & 0 & t_{\rho(i)} & \cdots \\ \rho(i{+}1) & & & & 0 & t_{\rho(i+1)} & 0 & 0 \end{array}$$

Now define M' to be equal to M everywhere except:

$$\begin{array}{c} \\ i \\ i+1 \\ \rho(i) \\ \rho(i+1) \end{array} \begin{array}{c} \begin{array}{ccccccc} i & i+1 & \rho(i) & \rho(i+1) & i+N & i+N+1 & \rho(i)+N \end{array} \\ \begin{bmatrix} 0 & 0 & t_{i+1} & 2t_i & \cdots & & \\ & 0 & 0 & -t_i & 0 & \cdots & \\ & & 0 & 0 & t_{\rho(i)} & 2t_{\rho(i)} & \cdots \\ & & & 0 & 0 & -t_{\rho(i+1)} & 0 \end{bmatrix} \end{array}$$

and P to be the identity matrix except

$$\begin{array}{c|cc} & i & i+1 \\ \hline i & 0 & -1 \\ i+1 & 1 & 2 \end{array}$$

Let us check that $M' \in X_\rho$. $M'^2 = 0$ by direct computation. Furthermore, the upper triangle of M' (the first four columms in the region of interest described above) has same non-zero entries as $\underline{\rho}$ except at the irrelevant entry $(i, \rho(i+1))$, so satisfies the same rank conditions as $\underline{\rho}$, so is in its B-orbit. And $(M'^2)_{i,i+N} = t_{i+1}t_{\rho(i)} = t_i t_{\rho(i+1)} = (M'^2)_{i+1,i+1+N}$. Thus, $M' \in F_\rho \cap \{M : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}\} \subset X_\rho$. One then simply computes, using $t_i t_{\rho(i+1)} = t_{i+1} t_{\rho(i)}$, that $PM'P = M$.

5.5. **From Brauer loop polynomials back to Joseph–Melnikov polynomials.** Finally, we comment on the connection between the results of section 5.1 concerning Joseph–Melnikov polynomials J_π and those of section 5.2 concerning Brauer loop polynomials Ψ_π. Recall that these two sets of multidegrees are related by Theorem 7. Thus taking the $B \to \infty$ limit in the equations (4.19,4.20) satisfied by the Ψ_π should result in equations satisfied by the J_π.

Let us first consider Eq. (4.19). For any link pattern π (i.e. involution without fixed point in even size), there are only two possibilities: either $f_i \cdot \pi$ has one more crossing than π, or one fewer. If it has one more crossing, then the right hand side is of lower degree in B than the left hand side, and we find that $\tilde\partial_i J_\pi = 0$. If it has one fewer crossing, then taking B-leading terms on both sides of the equation results in Eq. (5.4) with the identification $f_i \cdot \pi = \bar{f}_i^{-1} \cdot \pi$. (The meaning of this strange change of notation will be explained in the final paragraph.)

Next start from Eq. (4.20), valid for any link pattern π, and send B to infinity. In order to compare degrees in B of the various terms in the sum over π', $e_i \cdot \pi' = \pi$, we need to compute their number of crossings. But since e_i cannot create crossings, the number of crossings of π' is greater or equal to that of π. Thus, the B-leading term of Eq. (4.20) is exactly Eq. (5.3), in which one must sum over π' that are preimages of π and have the same number of crossings as π. This is equivalent to the prescription given in the text after Eq. (5.3).

Finally, from the algebraic point of view, note that $B \to \infty$ corresponds to the parameter β of the Brauer algebra being sent to the value 2. This is a degenerate situation in which the R-matrix given by Eq. (4.2) loses its term proportional to f_i, and so doing becomes the rational Temperley–Lieb R-matrix (see [DFZJ05b] for a related discussion of the Temperley–Lieb qKZ equation). This strongly suggests that a more interesting point of view is to replace f_i by $\bar{f}_i := (1 - \beta/2) f_i$ and only then take the limit $\beta \to 2$. The resulting algebra, the **degenerate Brauer ($\beta = 2$) algebra,** is given by generators e_i, $\bar{f}_i$,

$i = 1, \dots, N-1$ and relations

$$\begin{aligned}
&e_i^2 = 2e_i \qquad e_i e_{i\pm 1} e_i = e_i \\
&\bar{f}_i^2 = 0 \qquad \bar{f}_i \bar{f}_{i+1} \bar{f}_i = \bar{f}_{i+1} \bar{f}_i \bar{f}_{i+1} \\
&\bar{f}_i e_i = e_i \bar{f}_i = 0 \qquad \bar{f}_{i+1} \bar{f}_i e_{i+1} = \bar{f}_i \bar{f}_{i+1} e_i = 0 \\
&e_i e_j = e_j e_i \qquad \bar{f}_i \bar{f}_j = \bar{f}_i \bar{f}_j \qquad e_i \bar{f}_j = \bar{f}_j e_i \qquad |i-j| > 1
\end{aligned}$$

Its action on linear combinations of link patterns is the same as usual, with the additional rule that if a link pattern π is such that the arches coming out of i and $i+1$ cross, then $\bar{f}_i \cdot \pi = 0$. It now has a non-trivial R-matrix and this way, the various equations satisfied by the J_π are its qKZ equation. (This also explains the notation $\bar{f}_i^{-1} \cdot \pi$ used before, since $\bar{f}_i$, contrary to f_i, is not an involution or even invertible: by $\bar{f}_i^{-1} \cdot \pi$ we mean the unique preimage of π when it exists). Note in particular that the $\bar{f}_i$ generate a subalgebra called the nil-Hecke algebra, which was discussed in a similar context in [FK96].

Appendix A. The affine Weyl group $\hat{\mathcal{S}}_N$

The affine Weyl group $\hat{\mathcal{S}}_N$ is defined by generators f_i, $i \in \mathbb{Z}/N\mathbb{Z}$, and relations

$$f_i^2 = 1 \qquad (f_i f_{i+1})^3 = 1 \qquad f_i f_j = f_j f_i \qquad j \neq i-1, i+1 \tag{A.1}$$

It is a semi-direct product $\mathcal{S}_N \ltimes \mathbb{Z}^{N-1}$, as will be made explicit now.

First define an alternative description of $\hat{\mathcal{S}}_N$ which is particularly convenient for our purposes. Call $\star$ the canonical projection from $\mathbb{Z}$ to $\mathbb{Z}/N\mathbb{Z}$. Consider the group F_N of invertible maps ϕ from $\mathbb{Z}$ to $\mathbb{Z}$ such that $\phi(i+N) = \phi(i) + N$ for all $i \in \mathbb{Z}$, endowed with composition. (This group appeared also in [ER96], where it is the "group of juggling patterns with period N".) Then it is easy to show that there exists an injective morphism ι from $\hat{\mathcal{S}}_N$ to F_N such that

$$\iota(f_i) : j \mapsto \begin{cases} j & j^\star \neq i, i+1 \\ j+1 & j^\star = i \\ j-1 & j^\star = i+1 \end{cases} \tag{A.2}$$

Its image is precisely the maps $\phi \in F_N$ such that $\sum_{i=1}^{N} (\phi(i) - i) = 0$ ("juggling patterns with 0 balls"). From now on we identify $\hat{\mathcal{S}}_N$ with its image in F_N.

Next define the projection p from $\hat{\mathcal{S}}_N$ to $\mathcal{S}_N$ viewed as the group of permutations of $\mathbb{Z}/N\mathbb{Z}$. Any $\phi \in F_N$ has a unique factorization $\star\phi = p(\phi)\star$ with $p(\phi) \in \mathcal{S}_N$, and in particular by restriction we get a map $p : \hat{\mathcal{S}}_N \to \mathcal{S}_N$.

The kernel of p is made of maps ϕ such that $\phi(i) = i \mod N$ for all $i \in \mathbb{Z}$. Thus it is isomorphic to $(\mathbb{Z}^N, +)$ via $\phi \mapsto ((\phi(1)-1)/N, \dots, (\phi(N)-N)/N)$. Restricting to $\hat{\mathcal{S}}_N$ we obtain a subgroup $\{(k_1, \dots, k_N) : \sum_{i=1}^N k_i = 0\}$ of $\mathbb{Z}^N$ isomorphic to $\mathbb{Z}^{N-1}$.

It is now an easy exercise to conclude that $\hat{\mathcal{S}}_N \cong \mathcal{S}_N \ltimes \mathbb{Z}^{N-1}$. We may choose as a particular subgroup isomorphic to $\mathcal{S}_N$ the one generated by $f_1, \dots, f_{N-1}$.

In this paper, particular subsets $\hat{\mathcal{S}}_{\pi,\pi'}$ of $\hat{\mathcal{S}}_N$ are defined, see Eq. (4.12); here π and π' are two involutions of $\mathbb{Z}/N\mathbb{Z}$ without fixed points, on which elements s of $\hat{\mathcal{S}}_N$ act by conjugation by $p(s)$. Using our alternative description of $\hat{\mathcal{S}}_N$ we can find a much simpler characterization of $\hat{\mathcal{S}}_{\pi,\pi'}$:

Proposition 10.

$$\hat{\mathcal{S}}_{\pi,\pi'} = \{s \in \hat{\mathcal{S}}_N, s \cdot \pi = \pi' \,\big|\, \forall i, j \in \mathbb{Z}, \pi(i^\star) = j^\star \textit{ and } i < j \Rightarrow s(i) < s(j)\}$$

Proof. Let $s = f_{i_k} \cdots f_{i_1}$ be an element of $\hat{\mathcal{S}}_{\pi,\pi'}$ as in Eq. (4.12). We prove by induction on k that s satisfies the property of Prop. 10 (the induction is for all π' simultaneously). It is trivial at $k = 0$; to go from $k-1$ to k, write $s = f_{i_k} s'$ with $s' = f_{i_{k-1}} \cdots f_{i_1}$, and pick a pair of integers i, j with $\pi(i^\star) = j^\star$. Due to the induction hypothesis we know that $s'(i) < s'(j)$ and want to apply f_{i_k} to both sides of the inequality. Since the effect of f_{i_k} is only to increase/decrease by 1 (c.f. Eq. (A.2)), and it only affects i_k and $i_k + 1$, we have automatically $s(i) < s(j)$ unless $s'(i)^\star = i_k$ and $s'(j) = s'(i) + 1$. But this contradicts the defining property in Eq. (4.12) at $\ell = k$: indeed we would have $(f_{i_{k-1}} \cdots f_{i_1} \cdot \pi)(i_k) = (s' \cdot \pi)(i_k) = p(s')(\pi(p(s')^{-1}(i_k))) = p(s')(\pi(i^\star)) = p(s')(j^\star) = i_{k+1}$.

Conversely, assume s satisfies the property of Prop. 10, and write a decomposition $s = f_{i_k} \cdots f_{i_1}$ of *minimum length* k. We claim this word satisfies Eq. (4.12). To abbreviate let us denote $w_m = f_{i_m} \cdots f_{i_1}$, and assume there is a step ℓ such that $(w_{\ell-1} \cdot \pi)(i_\ell) = i_\ell + 1$. In other words there is a pair i, j such that $j^\star = \pi(i^\star)$ and $w_{\ell-1}(i) = i_\ell$, $w_{\ell-1}(j) = i_\ell + 1$ – hence also $w_\ell(i) = i_\ell + 1$, $w_\ell(j) = i_\ell$. Consider now $S = \{m = 1, \ldots, k : (w_{m-1}(i) - w_{m-1}(j))(w_m(i) - w_m(j)) < 0\}$. S is non-empty since $\ell \in S$. Furthermore, the property of Prop. 10 implies that $w_0(i) - w_0(j) = i - j$ and $w_k(i) - w_k(j) = s(i) - s(j)$ have same sign; therefore S has even cardinality. We may then pick a pair of distinct elements, say $\ell, \ell' \in S$ and remove them from the word: it is simple to check that the new word $f_{i_k} \cdots \widehat{f_{i_\ell}} \cdots \widehat{f_{i_{\ell'}}} \cdots f_{i_1}$ still equals s, which contradicts the hypothesis of minimum length of the original word. $\square$

We can now obtain the

Corollary. *(Lemma 7) $\hat{\mathcal{S}}_{\pi_0,\pi_0}$ is the subgroup of $\hat{\mathcal{S}}_N$ generated by the $f_i f_{i+n}$, $i \in \mathbb{Z}/N\mathbb{Z}$.*

Proof. That the $f_i f_{i+n}$ belong to $\hat{\mathcal{S}}_{\pi_0,\pi_0}$ is elementary.

Conversely, consider $s \in \hat{\mathcal{S}}_{\pi_0,\pi_0}$. By successive multiplications by $f_i f_{i+n}$ we want to reduce it to the identity. Apply Prop. 10:

$$\hat{\mathcal{S}}_{\pi_0,\pi_0} = \{s \in \hat{\mathcal{S}}_N, s \cdot \pi_0 = \pi_0 : \forall i \in \mathbb{Z}\ s(i) < s(i+n)\} \tag{A.3}$$

The first required property is that $p(s)$ commute with the involution π_0. We know that the group of such permutations is isomorphic to $\mathcal{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ where $\mathcal{S}_n$ permutes the n cycles and the $\mathbb{Z}/2\mathbb{Z}$ permute the two elements of each cycle. In this formulation, $f_i f_{i+n}$ can be viewed as the elementary transposition $(i, i+1)$ of $\mathcal{S}_n$. Therefore, by successive multiplications by $f_i f_{i+n}$ one can assume that $p(s)$ preserves each cycle of π_0, i.e. $s(i) = i \mod n$ for all i.

Define the integers $\tilde{k}_i = (s(i) - i)/n$. According to Eq. (A.3), $\tilde{k}_i < 1 + \tilde{k}_{i+n} < 2 + \tilde{k}_i$ and therefore $\tilde{k}_i = \tilde{k}_{i+n}$. Similarly as before, $s \mapsto (\tilde{k}_1, \ldots, \tilde{k}_n)$ provides an injective morphism from the $s \in \hat{\mathcal{S}}_{\pi_0,\pi_0}$ such that $p(s)$ preserves each cycle of π_0 to the n-uplets $(\tilde{k}_1, \ldots, \tilde{k}_n)$ such that $\sum_{i=1}^n \tilde{k}_i = 0$.

We finally multiply s by elements of the form $T_i = U_i U_{i+1}^{-1}$, $i = 1, \ldots, n$, where $U_i = f_{i+n} f_i\, f_{i+n-1} f_{i-1} \cdots f_{i+1} f_{i+n+1}\, f_i f_{i+n}$. The T_i also preserve each cycle of π_0 and correspond to the values $\tilde{k}_i = \tilde{k}_{i+n} = +1$, $\tilde{k}_{i+1} = \tilde{k}_{i+n+1} = -1$ and the other $\tilde{k}_j = 0$. This clearly allows to reduce to $\tilde{k}_i = 0$, i.e. $s = 1$. $\square$

Remark: In the alternative description of $\hat{\mathcal{S}}_N$ one could introduce the extra map $r : i \mapsto i+1$ ("the standard 1-ball juggling pattern"). This would be the proper abstract element corresponding to the operator r on V introduced in the text, such that $rf_ir^{-1} = f_{i+1}$.

Appendix B. More on X_ρ

In section 5.4 the following varieties are introduced

$$X_\rho = E_\rho \cap \left\{ M : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N} \right\}$$

In this appendix, ρ will always be a link pattern such that $\rho(i) \neq i+1$ and the chords coming out of i and $i+1$ *cross*.

Our main goal is to describe explicitly a dense subset of orbits inside X_ρ. This is slightly tricky because these orbits do not have the simple structure that is found in generic orbits of E_π (cf Prop. 3 of [KZJ07] and Lemma 17 below) i.e. they do not possess representatives which are permutation matrices. We recall that $U = \{M \in \mathcal{M}_N : M_{ii} = 1\}$.

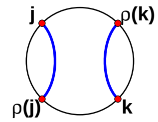
Lemma 15. *Consider matrices* $M \in E$ *of the form*

$$\begin{array}{c c}
 & \begin{array}{ccccccc} i & i{+}1 & \rho(i) & \rho(i{+}1) & i{+}N & i{+}N{+}1 & \rho(i){+}N \end{array} \\
\begin{array}{c} i \\ i{+}1 \\ \rho(i) \\ \rho(i{+}1) \end{array} &
\begin{bmatrix}
0 & 0 & t_i & 0 & \cdots & & \\
 & 0 & 0 & t_{i+1} & 0 & \cdots & \\
 & & 0 & 0 & t_{\rho(i)} & x & \cdots \\
 & & & 0 & 0 & t_{\rho(i+1)} & y
\end{bmatrix}
\end{array}$$

in the rows and columns for which we chose mod N *representatives of the form* $i < i+1 < \rho(i) < \rho(i+1) < i+N$*, where the parameters satisfy* $t_it_{\rho(i)} = t_{i+1}t_{\rho(i+1)} \neq 0$*; and whose other non-zero entries are* $M_{k,l} = t_k$ *where* $\rho(k) = l$ *and* $k < l < k+N$*. Then the union of their orbits by conjugation by* U *is a dense subset of* X_ρ*.*

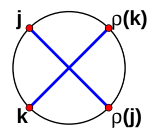
Proof. First consider the projection $(R, L) \mapsto R$ (we may assume by cyclic invariance $i = 1$, which means that on the picture of the lemma, the projection corresponds to keeping the first 4 columns of the matrix). The resulting matrix R has the same non-zero entries as $\rho_<$ and so is in $B \cdot \rho_<$. Furthermore $(M^2)_{i,i+N} = t_it_{\rho(i)}$ and $(M^2)_{i+1,i+1+N} = t_{i+1}t_{\rho(i+1)}$, so these two quantities are equal. The matrices of the lemma therefore belong to $F_\rho \cap \{M : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}\} \subset X_\rho$. X_ρ is by definition stable by conjugation by U, so their orbits sit inside X_ρ.

Next we compute the dimension of a single (generic) orbit. For that we consider the infinitesimal stabilizer of U on M of the form of the lemma. The equation $MP = PM$ (where P is strictly upper triangular) takes the same form as in Thm. 4 of [KZJ07], namely for each pair of chords of ρ we have associated equations: $\{j, \rho(j)\}$ and $\{k, \rho(k)\}$:

(1) The chords $\{j, \rho(j)\}$ and $\{k, \rho(k)\}$ do not cross each other:  , in which case we can choose representatives $j < \rho(j) < k < \rho(k) < j + N$ and we find:

$$
\begin{aligned}
t_j P_{jk} + t_k P_{\rho(j),\rho(k)} &= 0 \\
t_k P_{kj} + t_i P_{\rho(k),\rho(j)} &= 0 \\
t_{\rho(j)} P_{\rho(j),k} &= 0 \\
t_{\rho(k)} P_{\rho(k),j} &= 0
\end{aligned}
$$

(note that they form groups of two, related by a rotation of 180° or equivalently exchange of j and k). They are generically (for non-zero t's) non-trivial and independent from each other.

(2) The chords $\{j, \rho(j)\}$ and $\{k, \rho(k)\}$ cross each other:  , in which case we can choose representatives $j < k < \rho(j) < \rho(k) < j + N$ and we find

$$
\begin{aligned}
t_j P_{j,k} + t_k P_{\rho(j),\rho(k)} &= 0 \\
t_k P_{k,\rho(j)} + t_{\rho(j)} P_{\rho(k),j} &= 0 \\
t_{\rho(j)} P_{\rho(j)\rho(k)} + t_{\rho(k)} P_{j,k} &= 0 \\
t_{\rho(k)} P_{\rho(k)j} + t_j P_{k,\rho(j)} &= 0
\end{aligned}
$$

(all these equations are obtained from each other by rotation of 90°, which is the symmetry of the diagram). If $\{i, i + 1\} \not\subset \{j, k, \rho(j), \rho(k)\}$, then generically, $t_j t_{\rho(j)} \neq t_k t_{\rho(k)}$ and the linear system is non-degenerate, so that there are exactly four independent equations.

However, if $\{i, i+1\} \subset \{j, k, \rho(j), \rho(k)\}$ (note that by hypothesis the chords coming from $i, i + 1$ are crossing), then we have $t_i t_{\rho(i)} = t_{i+1} t_{\rho(i+1)}$ and the linear system becomes degenerate, so that we find only two independent equations.

The conclusion is that each pair of chords contributes 4 equations, except one of them that contributes 2, hence a total of $4 \times n(n - 1)/2 - 2 = 2(n^2 - n - 1)$. The dimension of an orbit is the dimension of the group minus the dimension of the stabilizer, which is precisely this number of equations, that is $2(n^2 - n - 1)$.

Next we check that each orbit possesses a unique representative of the form of the lemma. Write $MP = PM'$, $P \in U$. It is perhaps useful to write out explictly $MP - PM'$ in the rows and columns of interest:

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \rho(i) & \rho(i+1) & i+N & i+N+1 & \rho(i)+N \end{array} \\
\begin{array}{c} i \\ i+1 \\ \rho(i) \\ \rho(i+1) \end{array} &
\left[\begin{array}{ccccc}
t_i - t'_i & P_{\rho(i),\rho(i+1)} t_i - P_{i,i+1} t'_{i+1} & \cdots & & \\
0 & t_{i+1} - t'_{i+1} & P_{\rho(i+1),i} t_{i+1} - P_{i+1,\rho(i)} t'_{\rho(i)} & \cdots & \\
0 & 0 & t_{\rho(i)} - t'_{\rho(i)} & x - x' + P_{i,i+1} t_{\rho(i)} - P_{\rho(i),\rho(i+1)} t_{\rho(i+1)} & \cdots \\
 & 0 & 0 & t_{\rho(i+1)} - t'_{\rho(i+1)} & y - y' + P_{i+1,\rho(i)} t_{\rho(i+1)} - P_{\rho(i+1),i} t'_i
\end{array}\right]
\end{array}
$$

In fact the entry $(k,\rho(k))$ reads $t_k = t'_k$, for all k. Now if we use once again the relation $t_i t_{\rho(i)} = t_{i+1} t_{\rho(i+1)}$ we find that the remaining four non-trivial equations simplify so that $x = x'$, $y = y'$. So the representatives are unique.

Finally, the space of matrices of the form of the lemma is $2n+2$ parameters minus 1 equation, that is $2n+1$. So the union of orbits has dimension $2(n^2-n-1)+2n+1 = 2n^2-1 = \dim X_\rho$ and we conclude by irreducibility of X_ρ that it is dense. □

From this we immediately deduce

Lemma 16.

$$X_\rho \subseteq E_\pi \cap E_\rho \cap E_{f_i\cdot\rho}$$

where π and $f_i \cdot \rho$ are the link patterns obtained from ρ by "uncrossing" the chords coming from $i, i+1$ in the two possible ways.

Proof. $X_\rho \subset E_\rho$ by definition. The other two inclusions are obtained by checking them on the orbit representatives of lemma 15. From this point of view E_π and $E_{f_i\cdot\rho}$ play strictly identical roles, so we do the reasoning for E_π only. Consider the one-parameter family of matrices M_z which are equal to M in lemma 15 except:

$$\begin{array}{c|c}
 & \begin{array}{ccccccc} i & i{+}1 & \rho(i) & \rho(i{+}1) & i{+}N & i{+}N{+}1 & \rho(i){+}N \end{array} \\
\begin{array}{c} i \\ i{+}1 \\ \rho(i) \\ \rho(i{+}1) \end{array} &
\begin{bmatrix}
0 & t_i z & t_i & 0 & \cdots & & \\
 & 0 & 0 & t_{i+1} & 0 & \cdots & \\
 & & 0 & -t_{i+1} z & t_{\rho(i)} & x & \cdots \\
 & & & 0 & 0 & t_{\rho(i+1)} & y
\end{bmatrix}
\end{array}$$

Using once more $t_i t_{\rho(i)} = t_{i+1} t_{\rho(i+1)}$, we check that $M_z^2 = 0$. For $z = 0$ we recover the matrix of the lemma. Furthermore,

$$\begin{aligned}
(M_z^2)_{i,i+N} &= t_i t_{\rho(i)} \\
(M_z^2)_{i+1,i+N+1} &= t_{i+1} t_{\rho(i+1)} \\
(M_z^2)_{\rho(i),\rho(i)+N} &= t_i t_{\rho(i)} - yzt_{i+1} \\
(M_z^2)_{\rho(i+1),\rho(i+1)+N} &= t_{i+1} t_{\rho(i+1)} - yzt_{i+1}
\end{aligned}$$

so for generic z, y, t_k we have the following coincidences of $(M_z^2)_{k,k+N}$ (and no other): $(M_z^2)_{i,i+N} = (M_z^2)_{i+1,i+1+N}$, $(M_z^2)_{\rho(i),\rho(i)+N} = (M_z^2)_{\rho(i+1),\rho(i+1)+N}$ and $(M_z^2)_{k,k+N} = (M_z^2)_{\rho(k),\rho(k)+N}$ for $k \neq i, i+1, \rho(i), \rho(i+1)$. These are exactly the chords of π. According to Thm. 1, this implies that $M_z \in E_\pi$. Since E_π is closed, $M_0 \in E_\pi$. □

In fact, it is a consequence of the conjecture that equations of Thm. 2 define E_π, that X_ρ is equal to the intersection of any pairs of the three components $E_\pi, E_\rho, E_{f_i\cdot\rho}$.

We now briefly discuss $X_{f_i\cdot\rho}$. Recall that Prop. 9 says that it has two components, one being X_ρ and the other one called Y_ρ.

Lemma 17. *The irreducible set $U \cdot \{\underline{f_i \cdot \rho}\, t : t \in T$ and $t_i t_{\rho(i+1)} = t_{i+1} t_{\rho(i)}\}$ is dense in Y_ρ.*

Proof. First note that

$\{\underline{f_i \cdot \rho}\, t : t \in T \text{ and } t_i t_{\rho(i+1)} = t_{i+1} t_{\rho(i)}\} \subseteq F_{f_i\cdot\rho} \cap \{M : (M^2)_{i,i+N} = (M^2)_{i+1,i+1+N}\} = X_{f_i\cdot\rho} \cap F_{f_i\cdot\rho}$

so that it is a subset of Y_ρ. The rest of the proof is strictly identical to that of Prop. 3 of [KZJ07] (or of Lemma 15 but without the added complication of "irregular" orbits) and we shall only sketch it here.

We first compute the dimension of a single orbit via that of the infinitesimal stabilizer of U on $\underline{f_i \cdot \rho}\, t$ and find $2n(n-1)$. We then check that each U-orbit contains a unique element of the form $\underline{f_i \cdot \rho}\, t$. Finally, we compute $\dim\{\underline{f_i \cdot \rho}\, t, t \in T \text{ and } t_i t_{\rho(i)} = t_{i+1} t_{\rho(i+1)}\} = 2n-1$ and find that the total dimension is $2n(n-1) + 2n - 1 = 2n^2 - 1 = \dim Y_\rho$. We conclude by irreducibility of Y_ρ. $\square$

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